

Common Agency and Coordinated Bids in Sponsored Search Auctions

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Abstract

The transition of the advertisement market from traditional media to the internet has induced a proliferation of marketing agencies specialized in bidding in the auctions used to sell advertisement space on the web. We analyze how bidding delegation to a common marketing agency undermines both revenues and efficiency of the generalized second price auction, the format used by Google and Microsoft-Yahoo!. We characterize losses relative to the case of both full competition and agency-coordination under an alternative auction format (VCG mechanism). We propose a criterion to detect bid coordination and apply it to data from a major search engine.

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1 Introduction

In two influential papers, Edelman, Ostrovsky and Schwarz (2007, EOS hereafter) and Varian (2007) pioneered the study of the Generalized Second Price (GSP) auction, the mechanism used to allocate advertisement space on the results web page of search engines like Google, Microsoft Bing and Yahoo!.¹ These auctions represent one of the fastest growing and most economically relevant forms of online advertisement, with an annual growth of 10% and a total value of 50 billion dollars in 2013 (PwC, 2015). A recent trend in this industry, however, has the potential to alter the functioning of these auctions – and, hence, the profits in this industry – in ways that are not accounted for by the existing models. In particular, an increasing number of bidders are delegating their bidding campaigns to specialized Search Engine Marketing Agencies (SEMAs).² As a result, SEMAs often bid in the same auction on behalf of different advertisers. But this clearly changes the strategic interaction, as SEMAs have the opportunity to lower their payments by coordinating the bids of their clients.

In this paper we explore the impact of bidding delegation to a common SEMA on the performance of the GSP auction. Our theoretical analysis uncovers a striking fragility of this mechanism to the possibility of bid coordination. This is underscored by our finding that the GSP auction is outperformed, both in terms of revenues and efficiency, by a benchmark mechanism (the VCG) which is known to perform poorly under coordinated bidding. Further support to our theoretical results is provided by data of a major search engine, which we use to assess the revenue losses due to the presence of a SEMA. Overall, our findings suggest that the diffusion of SEMAs is likely to have a substantial impact on this market.

Studying bidding coordination in the GSP auction presents a number of difficulties, which are only partly due to the inherent complexity of the mechanism. An insightful analysis of the

¹Varian (2007) and EOS study a complete information environment. More recent work by Athey and Nekipelov (2012) maintains common knowledge of valuations but introduces uncertainty on the set of bidders and quality scores. Gomes and Sweeney (2014) instead study an independent private values model. In related work on the ad exchanges auctions, Balseiro, Besbes and Weintraub (2015) study the dynamic interaction among advertisers under limited budgets constraints.

²A survey by the Association of National Advertisers (ANA) among 74 large U.S. advertisers indicates that about 77% of the respondents fully outsource their search engine marketing activities (and 16% partially outsource them) to specialized agencies, see ANA (2011). Analogously, a different survey of 325 mid-size advertisers by Econsultancy (EC) reveals that the fraction of companies not performing their paid-search marketing in house increased from 53% to 62% between 2010 and 2011, see EC (2011).

problem thus requires a careful balance between tractability and realism of the assumptions.

For instance, since SEMAs in this market operate side by side with independent advertisers, it is important to have a model in which coordinated and competitive bidding coexist. But the problem of ‘partial cartels’ is acknowledged as a major difficulty in the literature (e.g., Hendricks, Porter and Tan (2008)).³ To address this problem, our model combines elements of cooperative and non-cooperative game theory, in the spirit of the seminal work of Ray and Vohra (1997, 2014) (see also Ray, 2008). We thus introduce a general notion of ‘Recursively-Stable Agency Equilibrium’ (RAE) which involves both equilibrium and stability restrictions, and which provides a unified framework to study the impact of SEMAs under different mechanisms.⁴ Second, it is well-known that strategic behavior in the GSP auction is complex and gives rise to a plethora of equilibria (Borgers, Cox, Pesendorfer and Petricek (2013)). Introducing a tractable and insightful refinement has been a key contribution of EOS and Varian (2007). But their refinements are not defined in a context in which some advertisers coordinate their bids. Thus, a second challenge we face is to develop a refinement for the model with SEMA, which ensures both tractability as well as clear economic insights.

More precisely, we modify EOS and Varian’s baseline model by introducing a SEMA, which we model as a player that chooses the bids of its clients in order to maximize the total surplus. Bidders that do not belong to the SEMA are referred to as ‘independents’, and place bids in order to maximize their own profits. To ensure a meaningful comparison with EOS competitive benchmark, and to avoid the severe multiplicity of equilibria in the GSP auction, we introduce a refinement of bidders’ best responses that distills the individual-level underpinnings of EOS ‘lowest-revenue envy-free’ equilibrium, and assume that independents place their bids accordingly. This device enables us to maintain the logic of EOS refinement for the independent bidders, even if EOS equilibrium is not defined in the game with SEMA. The SEMA in turn makes a proposal of a certain profile of bids to its clients. The proposal is implemented if it is ‘recursively stable’ in the sense that, anticipating the bidding strategies

³The literature on ‘bidding rings’, for instance, has either considered mechanisms in which non-cooperative behavior is straightforward (e.g., second price auctions with private values, as in Mailath and Zemski (1991)), or assumes that the coalition includes all bidders in the auction (as in the first price auctions of McAfee and McMillan (1992) and Hendricks et al. (2008)). See section 3.2 for a discussion of this literature.

⁴Apart from the common underlying philosophy, our notion of RAE differs from Ray and Vohra’s concepts in important ways, extensively discussed in section 3.2.

of others, and taking into account the possible unraveling of the rest of the coalition, no client has an incentive to abandon the SEMA and bid as an independent. Thus, similar to Ray and Vohra (1997, 2014), the outside options of the members of a coalition are equilibrium objects themselves, and implicitly incorporate the restrictions entailed by the underlying coalition formation game.

Within this general framework, we consider different models of coordinated bidding, in which the SEMA operates under different constraints. We first assume that the agency is constrained to placing bids which cannot be detected as ‘coordinated’ by an external observer. This is a useful working hypothesis, which also has obvious intrinsic interest. Under this assumption, we show that the resulting allocation in the GSP with SEMA is efficient and the revenues are the same as those that would be generated if the same coalition structure was bidding in a VCG auction. We then relax this ‘undetectability constraint’, and show that in this case the search engine revenues in the GSP auction are never higher, and are in fact typically lower, than those obtained in the VCG mechanism with coordination. Furthermore, once the ‘undetectability constraint’ is lifted, efficiency in the allocation of bidders to slots is no longer guaranteed by the GSP mechanism. Since the VCG is famously regarded as a poor mechanism under coordinated bidding, finding that it outperforms the GSP both in terms of revenues and allocation is remarkably negative for the GSP. Finally, to assess the extent to which the poor revenue performance of the GSP with coordinated bidding is due to the allocative distortion, we consider a benchmark in which the SEMA operates under the constraint of not altering the allocation of the competitive benchmark (which is efficient). We show that, even with this allocative restriction, the revenues are lower in the GSP auction than in the VCG mechanism.

This fragility of the GSP auction is due to the complex equilibrium effects it induces. In particular, by manipulating the bids of its members, in equilibrium the SEMA also affects the bids of the independents. The SEMA therefore has both a *direct* and an *indirect effect* in the GSP auction, and hence even a small coalition may have a large impact on total revenues. Depending on the structure of the SEMA, the indirect effect may be first order. As we will explain, our analysis uncovers that this is especially the case if the SEMA includes members which occupy low or adjacent positions in the ranking of valuations.

In the final part of the paper, we supplement our theoretical analysis applying it to search auctions data. First, we illustrate how our theoretical results can be used to detect potentially coordinated behavior, and to choose over alternative models of coordinated bidding. Second, we use the equilibrium restrictions to infer bounds on bidders' valuations, which in turn can be used to compute the counterfactual revenues of the search engine under competitive bidding. We show that, although the typical coalition in our dataset only has 2 bidders, coordination has economically sizable effects, up to a 7.9% average revenue loss across the keywords for which we detect coordination. Consistent with our theoretical analysis, the *indirect effect* drives most of the revenue losses in the data (75% on average).

The main implication of this study is that, under the current trend towards increased agency bidding, revenue sharing and allocative properties of the GSP auction will change. This is obviously interesting from a market design perspective. Although the design of sponsored search auctions has received considerable attention in the literature (e.g., Edelman and Schwarz (2010) and Celis et al. (2015)), this is the first study to point to the role of agencies and, importantly, to develop a methodology capable of accommodating the joint presence of competitive and coordinated bidding. Our results are also potentially relevant from an antitrust perspective. In particular, our characterization of the agency behavior is analogous to that of buying consortia, which have been sanctioned in the past.⁵ Nevertheless, the specificities of the market suggest a more nuanced view of the harm to consumers. We discuss this point and other policy implications in the concluding section.

Finally, from a broader perspective, we note that our findings complement the recent work of Einav et al. (2014), who also document a decline in the importance of individual bidding in auctions: they show that consumers are progressively shifting away from online auction platforms towards purchasing at posted prices. Hence, there is a sense in which bidding delegation – the focus of our paper – can be seen as the counterpart, on the advertisers' side, of the same phenomenon documented by Einav et al. (2014). Altogether, it emerges the picture of a sizeable market undergoing important transformations. Our focus on a specific, albeit important, aspect of this change should thus be complemented by further research.

⁵See, for instance, the case of the tobacco manufacturers consortium buying in the tobacco leaves auctions, *United States v. American Tobacco Company*, 221 U.S. 106 (1911).

2 The GSP auction

In this section we introduce the GSP auction and the necessary notation. The only original result in this section is Lemma 1, which shows that EOS *lowest-revenue envy-free* (LREF) equilibrium – originally defined as a refinement of the Nash equilibrium correspondence – can be equivalently defined as the fixed point of a refinement of individuals’ best responses. This result will play an important role in the next sections. That is because, by distilling the individual underpinnings of EOS refinement, it enables us to extend EOS approach to the analysis of the GSP auction with SEMA, in which LREF equilibrium is not defined.

Consider the problem of assigning agents $i \in I = \{1, \dots, n\}$ to slots $s = 1, \dots, S$, where $n \geq S$. In our case, agents are advertisers, and slots are positions for ads in the page for a given keyword. Slot $s = 1$ corresponds to the highest position, and so on until $s = S$, which is the slot in the lowest position. For each s , we let x^s denote the ‘click-through-rate’ (CTR) of slot s , that is the number of clicks that an ad in position s is expected to receive, and assume that $x^1 > x^2 > \dots > x^S > 0$. We also let $x^t = 0$ for all $t > S$. Finally, we let v_i denote the per-click-valuation of advertiser i , and we label advertisers so that $v_1 > v_2 > \dots > v_n$.

In the GSP auction, each advertiser submits a bid $b_i \in \mathbb{R}_+$. The advertiser who submits the highest bid obtains the first slot and pays a price equal to the second highest bid every time his ad is clicked; the advertiser with the second highest bid obtains the second slot and pays a price-per-click equal to the third highest bid, and so on. We denote bid profiles by $b = (b_i)_{i=1, \dots, n}$ and $b_{-i} = (b_j)_{j \neq i}$. For any profile b , we let $\rho(i; b)$ denote the rank of i ’s bid in b (ties are broken according to bidders’ labels).⁶ When b is clear from the context, we omit it and write simply $\rho(i)$. For any $t = 1, \dots, n$ and b or b_{-i} , we let b^t and b_{-i}^t denote the t -highest component of the vectors b and b_{-i} , respectively.

The rules of the auction are formalized as follows. For any b , if $\rho(i) \leq S$ bidder i obtains position $\rho(i)$ at price-per-click $p^{\rho(i)} = b^{\rho(i)+1}$. If $\rho(i) > S$, bidder i obtains no position.⁷ We maintain throughout that advertisers’ preferences, the CTRs and the rules of the auction are

⁶Formally, $\rho(i; b) := |\{j : b_j > b_i\} \cup \{j : b_j = b_i \text{ and } j < i\}| + 1$. This tie-breaking rule is convenient for the analysis of coordinated bidding. It can be relaxed at the cost of added technicalities (see footnote 17).

⁷In reality, bidders allocation to slots is determined adjusting advertisers’ bids by some ‘quality scores’. To avoid unnecessary complications, we only introduce quality scores in section 5 (see also Varian (2007)).

common knowledge. We thus model the GSP auction as a game $\mathcal{G}(v) = (A_i, u_i^{\mathcal{G}})_{i=1, \dots, n}$ where $A_i = \mathbb{R}_+$ denotes the set of actions of player i (his bids), and payoff functions are such that, for every i and every $b \in \mathbb{R}_+^n$, $u_i^{\mathcal{G}}(b) = (v_i - b^{\rho(i)+1}) x^{\rho(i)}$. Although it may seem unrealistic, EOS complete information assumption has been shown to be an effective modeling proxy (e.g., Athey and Nekipelov (2012), Che, Choi and Kim (2013) and Varian (2007)).

Note that any generic profile $b_{-i} = (b_j)_{j \neq i}$ partitions the space of i 's bids into $S + 1$ intervals. The only payoff relevant component of i 's choice is in which of these intervals he should place his own bid: any two bids placed in the same interval would grant bidder i the same position at the same per-click price (equal to the highest bid placed below b_i). So, for each $b_{-i} \in \mathbb{R}_+^{n-1}$, let $\pi_i(b_{-i})$ denote i 's favorite position, given b_{-i} .⁸ Then, i 's best-response to b_{-i} is the interval $BR_i(b_{-i}) = (b_{-i}^{\pi_i(b_{-i})}, b_{-i}^{\pi_i(b_{-i})-1})$. This defines the best-response correspondence $BR_i : \mathbb{R}_+^{n-1} \rightrightarrows \mathbb{R}_+$, whose fixed points are the set of (pure) Nash equilibria: $\mathcal{EG}^0(v) := \{b \in \mathbb{R}_+^n : b_i \in BR_i(b_{-i}) \forall i \in I\}$.

It is well-known that the GSP auction admits a multiplicity of equilibria (Borgers et al. (2013)). For this reason, EOS introduced a refinement of the set $\mathcal{EG}^0(v)$, the *LREF equilibrium*. As anticipated above, we consider instead a refinement of individuals' best response correspondence: for any $b_{-i} \in \mathbb{R}_+^{n-1}$, let

$$BR_i^*(b_{-i}) = \left\{ b_i^* \in BR_i(b_{-i}) : (v_i - b_{-i}^{\pi_i(b_{-i})}) x^{\pi_i(b_{-i})} = (v_i - b_i^*) x^{\pi_i(b_{-i})-1} \right\}. \quad (1)$$

In words, of the many $b_i \in BR_i(b_{-i})$ that would grant player i his favorite position $\pi_i(b_{-i})$, he chooses the bid b_i^* that makes him indifferent between occupying the current position and climbing up one position paying a price equal to b_i^* . The set of fixed points of the BR_i^* correspondence are denoted as $\mathcal{EG}(v) = \{b \in \mathbb{R}_+^n : b_i \in BR_i^*(b_{-i}) \text{ for all } i \in I\}$.

Lemma 1. *For any $b \in \mathcal{EG}(v)$, $b_1 > b_2$, $b_i = v_i$ for all $i > S$, and for all $i = 2, \dots, S$,*

$$b_i = v_i - \frac{x^i}{x^{i-1}} (v_i - b_{i+1}). \quad (2)$$

Hence, the fixed points of the BR^ correspondence coincide with EOS' LREF equilibria.*

⁸ Allowing ties in individuals' bids or non-generic indifferences complicates the notation, without affecting the results and the main insights. We thus ignore these issues here and leave the details to Appendix A.1.

In section 4 we will assume that independents in the GSP auction bid according to BR_i^* , both with and without the agency. Since, by Lemma 1, this is precisely the same assumption on individuals' behavior that underlies EOS' analysis, our approach ensures a meaningful comparison with the competitive benchmark. Lemma 1 obviously implies that our formulation inherits the many theoretical arguments in support of EOS refinement (e.g. EOS, Edelman and Schwarz (2010), Milgrom and Mollner (2014)). But it is also important to stress that, independent of equilibrium restrictions, this individual-level refinement is particularly compelling because it conforms to the tutorials on how to bid in these auctions provided by the search engines.⁹ This formulation is thus preferable from both a conceptual and a practical viewpoint.

Because of its well known properties, the VCG mechanism represents the standard benchmark in the literature on the GSP auction (e.g., EOS, Varian (2007) and Athey and Nekipelov (2012)). Given valuations v , the VCG mechanism is formally defined as a game $\mathcal{V}(v) = (A_i, u_i^{\mathcal{V}})_{i=1, \dots, n}$, where $A_i = \mathbb{R}_+$ and $u_i^{\mathcal{V}}(b) = v_i x^{\rho(i)} - \sum_{t=\rho(i)+1}^{S+1} b^t (x^{t-1} - x^t)$ for each i and $b \in \mathbb{R}_+^n$.

It is well-known that bidding $b_i = v_i$ is a dominant strategy in this game. In the resulting equilibrium, advertisers are efficiently assigned to positions. Furthermore, EOS (Theorem 1) showed that the position and payment of each advertiser in the dominant strategy equilibrium of the VCG are the same as in the LREF equilibrium of the GSP auction.

The next example will be used repeatedly throughout the paper to illustrate the relative performance of the GSP and VCG mechanism:

Example 1. *Consider an auction with four slots and five bidders, with the following valuations: $v = (5, 4, 3, 2, 1)$. The CTRs for the five positions are the following: $x = (20, 10, 5, 2, 0)$. In the VCG mechanism, bids are $b_i = v_i$ for every i , which induces total expected revenues of 96. Bids in the LREF of the GSP auction instead are as follows: $b_5 = 1$, $b_4 = 1.6$, $b_3 = 2.3$ and $b_2 = 3.15$. The highest bid b_1 is not uniquely determined, but it does not affect the revenues, which in this equilibrium are exactly the same as in the VCG mechanism: 96. Clearly, also the allocation is the same in the two mechanisms, and efficient.*

⁹See, for instance, the Google Ad Word tutorial in which Hal Varian teaches how to maximize profits by following this bidding strategy: <http://www.youtube.com/watch?v=jRx7AMb6rZ0>.

3 Search Engine Marketing Agencies

Our analysis of the Search Engine Marketing Agencies (SEMAs) focuses on their opportunity to coordinate the bids of different advertisers.¹⁰ We thus borrow the language of cooperative game theory and refer to the clients of the agency as ‘members of a coalition’ and to the remaining bidders as ‘independents’. To avoid unnecessary complications, we focus on environments that resemble what we observe in the data, and restrict our attention to cases with a single SEMA per auction.¹¹

We model the SEMA as a player that makes proposals of binding agreements to its members, subject to certain stability constraints. The independents then play the game which ensues from taking the bids of the agency as given. We assume that the agency seeks to maximize the coalition surplus, but is constrained to choosing proposals that are *stable* in two senses: first, they are consistent with the independents’ equilibrium behavior; second, no individual member of the coalition has an incentive to abandon it and play as an independent. We also assume that, when considering such deviations, coalition members are *farsighted* in the sense that they anticipate the impact of their deviation on both the independents and the remaining members of the coalition (cf. Ray and Vohra (1997), and Ray (2008)). The constraint for a coalition of size C thus depends on the solutions to the problems of all the subcoalitions of size $C - 1$. Therefore, the solution concept for the game with the agency will be defined recursively. We discuss the related literature in section 3.2.

In the next section we introduce a general definition of the ‘Recursively Stable Agency Equilibrium’ (RAE), which allows for arbitrary underlying mechanisms. This is useful in that it provides a general framework to analyze the impact of SEMAs under different mechanisms. We then specialize the analysis to the GSP and VCG mechanisms in section 4.

¹⁰Other complementary studies have focused on different aspects of marketing agencies, such as their role in attracting new customers and improving ad quality, but also to potentially hurt advertisers by leaking information to rivals that share the same agency (see Day (2014)). Here we abstract from all these important aspects, to focus on the strategic implications of coordinated bidding by a common agency.

¹¹Data from a major search engine reveal three stylized facts: (i) bidding delegation to agencies is pervasive; (ii) a common agency placing bids in a single auction on behalf of different advertisers occurs in an economically relevant amount of auctions; (iii) consistent with the agencies’ specialization, it is almost never the case that more than one agency bids on behalf of multiple bidders in the same auction.

3.1 The Recursively Stable Agency Equilibrium

Let $G(v) = (A_i, u_i^G)_{i=1, \dots, n}$ denote the baseline game (without a coalition) generated by the underlying mechanism (e.g., the GSP ($G = \mathcal{G}$) or the VCG ($G = \mathcal{V}$) mechanism), given the profile of valuations $v = (v_i)_{i \in I}$. For any $\mathcal{C} \subseteq I$ with $|\mathcal{C}| \geq 2$, we let \mathcal{C} denote the agency, and we refer to advertisers $i \in \mathcal{C}$ as ‘members of the coalition’ and to $i \in I \setminus \mathcal{C}$ as ‘independents’. The coalition chooses a vector of bids $b_{\mathcal{C}} = (b_j)_{j \in \mathcal{C}} \in \times_{j \in \mathcal{C}} A_j$. Given $b_{\mathcal{C}}$, the independents $i \in I \setminus \mathcal{C}$ simultaneously choose bids $b_i \in A_i$. We let $b_{-\mathcal{C}} := (b_j)_{j \in I \setminus \mathcal{C}}$ and $A_{-\mathcal{C}} := \times_{j \in I \setminus \mathcal{C}} A_j$. Finally, given profiles b or $b_{-\mathcal{C}}$, we let $b_{-i, -\mathcal{C}}$ denote the subprofile of bids of all independents other than i (that is, $b_{-i, -\mathcal{C}} := (b_j)_{j \in I \setminus \mathcal{C}: j \neq i}$).

We assume that the agency maximizes the sum of the payoffs of its members,¹² denoted by $u_{\mathcal{C}}(b) := \sum_{i \in \mathcal{C}} u_i(b)$, under three constraints. Two of these constraints are stability restrictions: one for the independents, and one for the members of the coalition. The third constraint, which we formalize as a set $R_{\mathcal{C}} \subseteq A_{\mathcal{C}}$, allows us to accommodate the possibility that the agency may exogenously discard certain bids (this restriction is vacuous if $R_{\mathcal{C}} = A_{\mathcal{C}}$). For instance, in section 4.2.1 we will consider the case of an agency whose primary concern is not being identified as inducing coordinated bids. In that case, $R_{\mathcal{C}}$ would be comprised of only those bids that are ‘undetectable’ to an external observer as coordinated. We denote the collection of exogenous restrictions for all possible coalitions as $\mathcal{R} = (R_{\mathcal{C}})_{\mathcal{C} \subseteq I: |\mathcal{C}| \geq 2}$.

Stability-1: The first stability restriction on the agency’s proposals is that they are stable with respect to the independents. For any $i \in I \setminus \mathcal{C}$, let $BR_i^G : A_{-i} \rightrightarrows A_i$ denote some refinement of i ’s best response correspondence in $G(v)$ (e.g., BR_i^* in $\mathcal{G}(v)$ or weak dominance in $\mathcal{V}(v)$). Define the *independents’ equilibrium correspondence* $BR_{-\mathcal{C}}^G : A_{\mathcal{C}} \rightrightarrows A_{-\mathcal{C}}$ as

$$BR_{-\mathcal{C}}^G(b_{\mathcal{C}}) = \{b_{-\mathcal{C}} \in A_{-\mathcal{C}} : \forall j \in I \setminus \mathcal{C}, b_j \in BR_j^G(b_{\mathcal{C}}, b_{-j, -\mathcal{C}})\}. \quad (3)$$

If the agency proposes a profile $b_{\mathcal{C}}$ that is not consistent with the equilibrium behavior

¹²This is a simplifying assumption, which can be justified in a number of ways. From a theoretical viewpoint, our environment satisfies the informational assumptions of Bernheim and Whinston (1985, 1986). Hence, as long as the SEMA is risk-neutral, this particular objective function may be the result of an underlying common agency problem. More relevant from an empirical viewpoint, the agency contracts most commonly used in this industry specify a lump-sum fee per advertiser and per campaign. Thus, the SEMA’s ability to generate surplus for its clients is an important determinant of its long run profitability.

of the independents (as specified by BR_{-c}^G), then that proposal does not induce a *stable agreement*. We thus incorporate this stability constraint into the decision problem of the agency, and assume that the agency can only choose bid profiles from the set

$$S_c = \{b_c \in A_c : \exists b_{-c} \text{ s.t. } b_{-c} \in BR_{-c}^G(b_c)\}. \quad (4)$$

Clearly, the strength of this constraint in general depends on the underlying game $G(v)$ and on the particular correspondence BR_{-c}^G that is chosen to model the independents' behavior. This restriction is conceptually important, and needed to develop a general framework for the analysis of coordinated bidding in arbitrary mechanisms. Nonetheless, the restriction plays no role in our results for the GSP and VCG mechanisms, because (4) will be either vacuous (Theorem 1) or a redundant constraint (Theorems 2 and 3).

Stability-2: When choosing bids b_c , the agency forms conjectures about how its bids would affect the bids of the independents. We let $\beta : S_c \rightarrow A_{-c}$ represent such conjectures of the agency. For any profile $b_c \in S_c$, $\beta(b_c)$ denotes the agency's belief about the independents' behavior, if she chooses profile b_c . It will be useful to define the set of conjectures β that are consistent with the independents playing an equilibrium:

$$B^* = \{\beta \in A_{-c}^{S_c} : \beta(b_c) \in BR_{-c}^G \text{ for all } b_c \in S_c\}. \quad (5)$$

The second condition for stability requires that, given the conjectures β , the members of the coalition have no incentives to leave the agency and start acting as independents. Hence, the outside option for coalition member $i \in \mathcal{C}$ is determined by the equilibrium outcomes of the game with coalition $\mathcal{C} \setminus \{i\}$. This constraint thus requires a recursive definition.

Let $E^{c'}(BR^G, \mathcal{R})$ denote the set of Recursively Stable Agency Equilibrium (RAE) outcomes for the game with coalition \mathcal{C}' (given restrictions \mathcal{R} and refinement BR^G). For coalitions of size $C = 1$ (that is, the coalition-less game $G(v)$), define the set of equilibria as

$$E^1(BR^G) = \{b \in \mathbb{R}_+^n : b_i \in BR_i^G(b_{-i}) \text{ for all } i \in I\}. \quad (6)$$

Now, suppose that $E^{c'}(BR^G, \mathcal{R})$ has been defined for all subcoalitions $\mathcal{C}' \subset \mathcal{C}$. For each

$i \in \mathcal{C}$, and for each $\mathcal{C}' \subseteq \mathcal{C} \setminus \{i\}$, define

$$\bar{u}_i^{\mathcal{C}'} = \begin{cases} \min_{b \in E^1(BR^G)} u_i(b) & \text{if } |\mathcal{C}'| = 1 \\ \min_{b \in E^{\mathcal{C}'}(BR^G, \mathcal{R})} u_i(b) & \text{if } |\mathcal{C}'| \geq 2 \end{cases}.$$

The set of RAE of the game with coalition \mathcal{C} is defined as follows:

Definition 1. A Recursively Stable Agency Equilibrium (RAE) of the game G with coalition \mathcal{C} , given restrictions \mathcal{R} and independents' equilibrium refinement BR^G , is a profile of bids and conjectures $(b^*, \beta^*) \in A_{\mathcal{C}} \times B^*$ such that:¹³

1. The independents play a mutual best response: for all $i \in I \setminus \mathcal{C}$, $b_i^* \in BR_i^G(b_{-i}^*)$.
2. The conjectures of the agency are correct: $\beta^*(b_{\mathcal{C}}^*) = b_{-\mathcal{C}}^*$.
3. The agency best responds to the conjectures β^* , given the exogenous restrictions (R) and the stability restrictions about the independents and the coalition members (S.1 and S.2, respectively):

$$\begin{aligned} b_{\mathcal{C}}^* &\in \arg \max_{b_{\mathcal{C}}} u_{\mathcal{C}}(b_{\mathcal{C}}, \beta^*(b_{\mathcal{C}})) \\ \text{subject to} &: (R) \ b_{\mathcal{C}} \in R_{\mathcal{C}} \\ &: (S.1) \ b_{\mathcal{C}} \in S_{\mathcal{C}} \\ &: (S.2) \ \text{for all } i \in \mathcal{C}, \ u_i(b_{\mathcal{C}}, \beta^*(b_{\mathcal{C}})) \geq \bar{u}_i^{\mathcal{C} \setminus \{i\}} \end{aligned}$$

The set of RAE outcomes for the game with coalition \mathcal{C} (given BR^G and $R_{\mathcal{C}}$) is:

$$E^{\mathcal{C}}(BR^G, \mathcal{R}) = \{b^* \in A : \exists \beta^* \text{ s.t. } (b^*, \beta^*) \text{ is a RAE}\}. \quad (7)$$

In section 4 we will apply the RAE to study the impact of a SEMA on the GSP auction, and compare it to the benchmark VCG mechanism. The RAE in the GSP and the VCG mechanism are obtained from Definition 1, once the game G , the correspondence BR^G and the exogenous restrictions \mathcal{R} are specified accordingly.

¹³Note that, by requiring $\beta^* \in B^*$, this equilibrium rules out the possibility that the coalition's bids are sustained by 'incredible' threats of the independents.

3.2 RAE: Discussion and Related Literature

Our notion of RAE is closely related to the ‘Equilibrium Binding Agreements’ of Ray and Vohra (1997, RV hereafter). Given a certain coalition structure, RV postulate that binding agreements are possible within a coalition. The objective is to endogenize the collection of agreements such that no subcoalition has an incentive to break the agreement and separate from the original coalition. Moreover, when considering such deviations, the subcoalition is ‘farsighted’ in the sense that it does not take the behavior of the other coalitions as given, nor does she assume that the remaining members of the coalition will band together. Instead, it tries to predict the coalition structure and the agreements that would ultimately arise as a result of its deviation. In equilibrium, such predictions are required to be correct. Because of the ‘farsightedness assumption’, RV’s equilibrium is defined recursively, as is our RAE.¹⁴

RV’s and our approach share the same fundamental philosophy. Like RV, we also maintain that binding agreements are only possible within the coalition, but the interaction between the agency and the independents, as well as among the independents, is fully non-cooperative. As in RV, the agency in our model is a proposer of a binding agreement, subject to certain stability constraints, which crucially incorporate RV’s farsightedness assumption. There are two main differences between our model and that in RV. On the one hand, our stability restriction (S.2) only allows the agency’s proposal to be blocked by individual members. In contrast, RV’s solution concept presumes that the coalition’s proposals may be blocked by the joint deviation of any set of members of the coalition. That advertisers can make binding agreements outside of the SEMA, and jointly block its proposals, seems unrealistic in our context. We thus depart from RV’s approach and maintain that both the interaction with the independents and the blocking of the agency’s proposals are fully non-cooperative. The second difference is in the underlying model of non-cooperative interaction. Being concerned with a general theory of coalition formation, RV adopt Nash equilibrium (this can be accommodated in our setting letting BR_i^G in Definition 1 coincide with the standard best-response correspondence). To analyze the VCG and GSP mechanisms, in contrast, we

¹⁴The idea of ‘farsightedness’ in coalition formation is further explored in Ray and Vohra (1997, 2014). For an application of this approach to the free-rider problem, see Ray and Vohra (2001). Ray (2008) and Ray and Vohra (2013) provide thorough discussions of the general approach. Aghion, Antras and Helpman (2007) have applied similar ideas to problems of international trade.

adopt refinements of the individuals’ best responses, hence the underlying non-cooperative solution concept is a refinement of Nash equilibrium.

The literature on bidding rings in auctions (e.g., Graham and Marshall (1987), Mailath and Zemski (1991), McAfee and McMillan (1992) Hendricks, Porter and Tan (2008)) also addresses related phenomena, but from a very different perspective. In particular, the main focus of this literature is on whether members of the coalition may be incentivized to share their private information. Maintaining EOS and Varian’s complete information setting, we implicitly abstract away the information extraction problem within the coalition. Furthermore, we don’t allow transfers between members of the coalition.¹⁵ More importantly, a key feature of our setting is the co-presence of coordinated and independent bidding. Combining cooperative and non-cooperative interaction is a well-known challenge in this literature, which either considers mechanisms in which non-cooperative behavior is straightforward (e.g., second price auction with private values), or assumes that the coalition includes all bidders in the auction (as in the first price auctions of McAfee and McMillan (1992) and Hendricks et al. (2008)). The notion of RAE enables us to combine cooperative and non-cooperative interaction in general mechanisms, even if non-cooperative behavior is complex.

4 Analysis

In the previous section we developed a machinery (the general notion of RAE) to study bid coordination in arbitrary mechanisms. In the following we apply it to the GSP auction and to the VCG mechanism, the traditional benchmark for the GSP auction in the literature.

Definition 2. *The RAE-outcomes in the GSP and VCG mechanism are defined as follows.*

Given constraints \mathcal{R} :

1. *The RAE of the GSP auction is obtained setting $G(v)$ and BR_i^G in Definition 1 equal to $\mathcal{G}(v)$ and to BR_i^* , respectively.*

¹⁵Allowing transfers would relax constraint (S.2) in the definition of RAE, and affect our results (for instance, it may induce inefficiencies even in the VCG mechanism, cf. Example 2). That different advertisers make side-payments to each other seems implausible in this market. If indirect transfers could be implemented through dynamic effects (e.g., swapping bids for some of its members) or across different keywords, distinct strategic issues might arise, which would best be studied considering a richer model.

2. The RAE of the VCG mechanism are obtained from Definition 1 letting $G(v) = \mathcal{V}(v)$ and letting BR_i^G be such that independents play the dominant (truthful) strategy.

For both mechanisms, we will refer to the case where \mathcal{R} is such that $R_{C'} = A_{C'}$ for all $C' \subseteq I$ as the ‘unconstrained’ case.¹⁶

We first present the analysis of RAE in the VCG mechanism (section 4.1), and then proceed to the analysis of the GSP auction (section 4.2). Our main conclusion is that, when a SEMA is present, the VCG mechanism outperforms the GSP auction both in terms of revenues and allocative efficiency. These results therefore uncover a striking fragility of the GSP mechanism with respect to the possibility of coordinated bidding.

In some of the results below we have that the equilibrium bid for some i is such that $b_i = b_{i+1}$. Since ties are broken according to bidders’ labels (cf. footnote 6), in this case bidder i obtains the position above $i + 1$. To emphasize this, we will write $b_i = b_{i+1}^+$.¹⁷

4.1 Coordinated Bidding in the VCG mechanism

As anticipated in Definition 2, we apply RAE to the game $\mathcal{V}(v)$, assuming that the independents play the dominant strategy (truthful bidding), and letting the exogenous restrictions \mathcal{R} be vacuous (i.e., such that $R_C = A_C$ for every $C \subseteq I$). It is easy to check that, under this specification of the independents’ equilibrium correspondence, the set $S_C = A_C$. Hence, constraint (S.1) in Definition 1 plays no role in the results of this subsection.

Theorem 1. *For any \mathcal{C} , the unconstrained RAE of the VCG is unique up to the bid of the highest coalition member. In this equilibrium, advertisers are assigned to positions efficiently ($\rho(i) = i$), independents’ bids are equal to their valuations and all the coalition members*

¹⁶Under this definition, the RAE-outcomes for the coalitions of size one (eq.6) coincide precisely with the non-agency equilibria introduced in section 2. Namely, the LREF (EOS) equilibria in the GSP and the dominant strategy equilibrium in the VCG.

¹⁷Without the tie-breaking rule embedded in ρ (footnote 6), the SEMA’s best replies may be empty valued. In that case, our analysis would go through assuming that SEMA’s bids are placed from an arbitrarily fine discrete bid (i.e., $A_C = (\mathbb{R}_+ \cap \varepsilon\mathbb{Z})^{|C|}$ where ε is the minimum bid increment). In such alternative model, the case $b_i = b_{i+1}^+$ can be thought of as i bidding the lowest feasible bid higher than b_{i+1} , i.e. $b_i = b_{i+1} + \varepsilon$. All our results would hold in such a discrete model, once the equilibrium bids in the theorems are interpreted as the limit of the equilibria in the discrete model, letting $\varepsilon \rightarrow 0$ (the notation b_{i+1}^+ is thus reminiscent of this alternative interpretation, as the right-hand limit $b_{i+1}^+ := \lim_{\varepsilon \rightarrow 0} (b_{i+1} + \varepsilon)$). Embedding the tie-breaking rule in ρ allows us to avoid these technicalities, and focus on the key economic insights.

(except possibly the highest) bid the lowest possible value that ensures their efficient position. Formally, in any RAE of the VCG mechanism, the bid profile \hat{b} is such that

$$\hat{b}_i \begin{cases} = v_i & \text{if } i \in I \setminus \mathcal{C}; \\ = \hat{b}_{i+1}^+ & \text{if } i \in \mathcal{C} \setminus \{\min(\mathcal{C})\} \text{ and } i \leq S; \\ \in (\hat{b}_{i+1}^+, v_{i-1}) & \text{if } i = \min(\mathcal{C}) \text{ and } i \leq S. \end{cases} \quad (8)$$

where we denote $v_0 := \infty$ and $\hat{b}_{n+1} := 0$.

The RAE of the VCG mechanism therefore are efficient, with generally lower revenues than in the VCG without a SEMA. Moreover, the presence of a SEMA has no impact on the bids of the independents (this follows from the strategy-proofness of the mechanism).

The efficiency result in Theorem 1 is due to the stability restrictions in RAE, which limits the SEMA's freedom to place bids. Restriction (S.2), in particular, requires that the agency's proposal gives no member of the coalition an incentive to abandon it and bid as an independent. A recursive argument further shows that the payoff that any coalition member can attain from abandoning the coalition is bounded below by the equilibrium payoffs in the baseline (coalition-less) game, in which assignments are efficient. The 'Pigouvian' logic of the VCG payments in turn implies that such (recursive) participation constraints can only be satisfied by the efficient assignment of positions. It is interesting to notice that the recursive stability restriction is key to this result. As shown by the next example, without the recursive stability restriction (S.2), inefficiency is possible in the VCG with bid coordination:

Example 2. Let $n = 5$, $v = (40, 25, 20, 10, 9)$. The CTRs for the five positions are $x = \{20, 10, 9, 1, 0\}$, and suppose that $\mathcal{C} = \{1, 2, 5\}$. If 2 remains in the coalition and keeps his efficient position, RAE-bids are $\hat{b} = (\hat{b}_1, 20^+, 20, 10, 0)$, the coalition's payoff is 650, and 2 obtains 150 (again, \hat{b}_1 is not pinned down, but revenues and payoffs are). If 2 were to stay in, but drop one position down, the bids would be $\hat{b} = (\hat{b}_1, 10^+, 20, 10, 0)$, and payoffs for the coalition and 2 would be equal to 655 and 145, respectively. If 2 abandons the coalition, the bids in the game in which $\mathcal{C}' = \{1, 5\}$ are $\hat{b} = (\hat{b}_1, 25, 20, 10, 0)$, and 2 obtains a payoff of 150. Thus, the coalition would benefit from lowering 2's position, but the recursive stability

condition does not allow such a move. Note also that the recursive definition matters: If the outside option were defined by the case with no coalition at all, 2 would not drop out even when forced to take the lower position, since 2's payoff in that case would be $141 < 145$.

Whereas the presence of a SEMA does not alter the allocation of the VCG mechanism, it does affect its revenues: in any RAE of the VCG mechanism, the SEMA lowers the bids of its members (except possibly the one with the highest valuation) as much as possible, within the constraints posed by the efficient ranking of bids. Since, in the VCG mechanism, lowering the i -th bid affects the price paid for all slots $s = 1, \dots, \min\{S + 1, i - 1\}$, even a small coalition can have a significant impact on the total revenues.

Example 3. Consider the environment in Example 1, and suppose that $\mathcal{C} = \{1, 3\}$. Then, the RAE of the VCG mechanism is $\hat{b} = (\hat{b}_1, 4, 2^+, 2, 1)$. The resulting revenues are 86, as opposed to 96 of the non-agency case.

4.2 Coordinated Bidding in the GSP auction

We turn next to the GSP auction. According to the refinement of the best responses introduced above, we set BR_i^G in equation (3) equal to the marginally envy-free best response correspondence (eq. 1). The resulting correspondence $BR_{-\mathcal{C}}^G$ therefore assigns, to each profile $b_{\mathcal{C}}$ in the set of exogenous restrictions $R_{\mathcal{C}}$, the set of independent profiles that are fixed points of the BR_i^* correspondence for all $i \notin \mathcal{C}$.

We begin our analysis characterizing the RAE when the agency is constrained to placing bid profiles that could not be detected as ‘coordinated’ by an external observer (the ‘undetectable coordination’ restriction). Theorem 2 shows that the equilibrium outcomes of the GSP with this restriction are exactly the same as the unrestricted RAE of the VCG mechanism. We consider this problem mainly for analytical convenience, but the result of Theorem 2 also has independent interest, in that it characterizes the equilibria in a market in which ‘not being detectable as a bid coordinator’ is a primary concern of the SEMA.

We lift the ‘undetectable coordination’ restriction in section 4.2.2. We show that, unlike the VCG mechanism, the unrestricted RAE of the GSP auction in general may be inefficient and induce strictly lower revenues than their counterparts in the VCG mechanism. In light of

the efficiency of the VCG mechanism we established earlier (Theorem 1), it may be tempting to impute the lower revenues of the GSP auction to the inefficiencies that it may generate. To address this question, in section 4.2.2 we also consider the RAE of the GSP auction when the agency is constrained to inducing efficient allocations. With this restriction, we show that the equilibrium revenues in the GSP auction are always lower than in the VCG mechanism (Theorem 3). The revenue ranking therefore is not a consequence of the allocative effects.

4.2.1 The ‘Undetectable Coordination’ Restriction: A VCG-Equivalence Result

Consider the following set of exogenous restrictions: for any $\mathcal{C} \subseteq I$ s.t. $|\mathcal{C}| > 1$,

$$R_{\mathcal{C}}^{UC} := \left\{ b_{\mathcal{C}} \in A_{\mathcal{C}} : \exists v' \in \mathbb{R}_+^{|\mathcal{C}|}, b_{-C} \in \mathbb{R}_+^{n-|\mathcal{C}|} \text{ s.t. } (b_{\mathcal{C}}, b_{-C}) \in \mathcal{EG}(v'_{\mathcal{C}}, v_{-C}) \right\}.$$

These exogenous restrictions have a clear interpretation: the set $R_{\mathcal{C}}^{UC}$ is comprised of all bid profiles of the agency that could be observed as part of a LREF equilibrium in the GSP auction without the agency, given the valuations of the independents $v_{-C} = (v_j)_{j \in I \setminus \mathcal{C}}$. For instance, consider a SEMA whose primary interest is not being detectable as inducing bid coordination by an external observer. The external observer (e.g., the search engine or the anti-trust authority) can only observe the bid profile, but not the valuations $(v_i)_{i \in \mathcal{C}}$. Then, $R_{\mathcal{C}}^{UC}$ characterizes the bid profiles that ensure the agency would not be detected as inducing coordination, even if the independents had revealed their own valuations. The next result characterizes the RAE of the GSP auction under these restrictions, and shows its revenue and allocative equivalence to the unrestricted RAE of the VCG mechanism:

Theorem 2. *For any \mathcal{C} , in any RAE of the GSP auction under the ‘undetectable coordination’ (UC) restriction, the bids profile \hat{b} is unique up to the highest bid of the coalition and up to the highest overall bid. In particular, let $v_{n+1}^f = 0$, and for each $i = n, \dots, 1$, recursively define $v_i^f := v_{i+1}^f$ if $i \in \mathcal{C}$ and $v_i^f = v_i$ if $i \notin \mathcal{C}$. Then, for every i ,*

$$\hat{b}_i \begin{cases} = v_i^f - \frac{x^i}{x^i - 1} (v_i^f - \hat{b}_{i+1}), & \text{if } i \neq 1 \text{ and } i \neq \min(\mathcal{C}); \\ \in \left[v_i^f - \frac{x^i}{x^i - 1} (v_i^f - \hat{b}_{i+1}), \hat{b}_{i-1} \right) & \text{otherwise} \end{cases}, \quad (9)$$

where $\hat{b}_0 := \infty$ and $x^i/x^{i-1} := 0$ whenever $i > S$. Moreover, in each of these equilibria, advertisers are assigned to positions efficiently ($\rho(i) = i$), and advertisers' payments are the same as in the corresponding 'unrestricted RAE' of the VCG mechanism (Theorem 1).

Note that, in the equilibrium of (9), every bidder i other than the highest coalition member and the highest overall bidder bids as an independent with valuation v_i^f would bid in the baseline competitive model (first line of eq. 9). For the independent bidders ($i \notin \mathcal{C}$), such v_i^f coincides with the actual valuation v_i . For coalition members instead, $v_i^f \neq v_i$ is a 'feigned valuation'. Though notationally involved, the idea is simple and provides a clear insight on the SEMA's equilibrium behavior: intuitively, in order to satisfy the UC-restriction, the SEMA's bids for each of its members should mimic the behavior of an independent in the competitive benchmark, for some valuation. The SEMA's problem therefore boils down to 'choosing' a feigned valuation, and bid accordingly. The optimal choice of the feigned valuation is the one which, given others' bids, and the bidding strategy of an independent, induces the lowest bid consistent with i obtaining the i -th position in the competitive equilibrium of the model with feigned valuations, which is achieved by $v_i^f = v_{i+1}^f$. Note that the fact that bidder i cannot be forced to a lower position is not implicit in the UC-restriction, but the result of the equilibrium restrictions.¹⁸ The last line of (9) corresponds to the bid of the highest coalition member and the highest overall bidder, required to be placed in their efficient position. The logic of the equilibrium implies that it results in an efficient allocation. Moreover, these equilibria induce the same individual payments (hence total revenues) as the unrestricted RAE of the VCG mechanism.

To understand the implications of this equilibrium, notice that in the GSP auction, the i -th bid only affects the payment of the $(i - 1)$ -th bidder. Hence, the 'direct effect' of bids manipulation is weaker in the GSP than in the VCG mechanism, where the payments for all positions above i are affected. Unlike the VCG mechanism, however, manipulating the bid of coalition member i also has an 'indirect effect' on the bids of all the independents placed above i , who lower their bids according to the recursion in (9).

¹⁸The reason is similar to that discussed for Theorem 1, only here is more complicated due to the fact that, in the GSP auction, the bids of the SEMA alter the bids placed by the independents.

Example 4. Consider the environment of Example 3, with $\mathcal{C} = \{1, 3\}$. Then, the UC-RAE is $\hat{b} = (\hat{b}_1, 2.9, 1.8, 1.6, 1)$, which results in revenues 86. These are the same as in the VCG mechanism, and 10 less than in the non-agency case (Example 1). Note that the bid $\hat{b}_3 = 1.8$ obtains setting $v_3^f = v_4 = 2$, and then applying the same recursion as for the independents. Also note that the ‘direct effect’, due to the reduction in \hat{b}_3 , is only equal to $(b_3^{EOS} - \hat{b}_3) \cdot x_2 = 5$ (where b_3^{EOS} denotes 3’s bid in the non-agency benchmark). Thus, 50% of the revenue loss in this example is due to the SEMA’s ‘indirect effect’ on the independents.

Thus, despite the simplicity of the payment rule in the GSP auction, the equilibrium effects in (9) essentially replicate the complexity of the VCG payments: once the *direct* and *indirect effects* are combined, the resulting revenue loss is the same in the two mechanisms. This result also enables us to simplify the analysis of the impact of the SEMA on the GSP, by studying the comparative statics of the unconstrained RAE in the VCG mechanism. We thus obtain the simple qualitative insights for this complicated problem:

Remark 1. Both in the VCG mechanism and in the UC-RAE of the GSP auction:

1. Holding everything else constant, the revenue losses due to the SEMA increase with the differences $(x_{i-1} - x_i)$ associated to the SEMA’s clients $i \in \mathcal{C}$.
2. Holding $(x_s - x_{s+1})$ constant (i.e., if it is constant in s), the revenue losses due to the SEMA are larger if: (i) The coalition includes members that occupy a lower position in the ranking of valuations; (ii) The coalition includes members that occupy adjacent positions in the ranking of valuations; (iii) The difference in valuations between members of the coalition and the independents immediately below them (in the ranking of valuations) is larger.

Part 1 is immediate from Theorem 1 and the transfers of the VCG payment (section 2). Part 2 is also straightforward: point (i) follows from the fact that, for any given decrease of the bid of a single bidder, the total reduction in revenues of the VCG mechanism increases with the number of agents placed above him. Point (ii) is due to the fact that, for any $i \neq \min\{\mathcal{C}\}$, if $i+1$ also belongs to the coalition, then the SEMA can lower i ’s bid below v_{i+1} , still preserving an efficient allocation. Point (iii) follows because, the lower the valuations of the independents ranked below a member of the coalition, the more the SEMA has freedom to lower the bid of that member without violating the efficient ranking of bids (Theorem 1).

4.2.2 Lifting the UC-Restriction: Revenue Losses and Inefficiency

As discussed in section 4.1, the revenues generated by the VCG may be largely affected by the presence of a SEMA, even if it comprises a small number of members. Theorem 2 therefore already entails a fairly negative outlook on the sellers' revenues in the GSP auction when a SEMA is active, even when it cannot be detected as explicitly inducing any kind of 'coordinated bidding'. In this section we show that, when the undetectability constraint is lifted, a SEMA may induce larger revenue losses as well as inefficient allocations in the GSP auction. Before doing that, however, we first consider a weaker set of exogenous restrictions, which force the SEMA to induce efficient allocations. This is useful to isolate the price-reducing effect of bidding coordination separately from its potential allocative effect. Theorem 3 shows that, even with this restriction, the auctioneer's revenues are no higher than in the unrestricted equilibria of the VCG mechanism.

Formally, let $\mathcal{R}^{EFF} = \{R_C^{EFF}\}_{C \subseteq I: |C| \geq 2}$ be such that, for each non trivial coalition $C \subseteq I$,

$$R_C^{EFF} := \{b_C \in A_C : \exists b_{-C} \in BR_{-C}^*(b_C) \text{ s.t. } \rho(i; (b_C, b_{-C})) = i \ \forall i \in I\}.$$

Definition 3. *An efficiency-constrained RAE of the GSP auction is a RAE of the GSP auction where the exogenous restrictions are given by $\mathcal{R} = \mathcal{R}^{EFF}$ and the agency's conjectures β^* satisfy $\rho_i(b_C, \beta^*(b_C)) = i$ for all $b_C \in R_C^{EFF}$ and all $i \in I$.*

Theorem 3. *Efficiency-constrained RAE of the GSP auction exist; in any such RAE: (i) the agency's payoff is at least as high as in any RAE of the VCG mechanism, and (ii) the auctioneer's revenue is no higher than in the corresponding equilibrium of the VCG auction. Furthermore, there exist parameter values under which both orderings are strict.*

By imposing efficiency as an exogenous constraint, Theorem 3 shows that the structure of the payments in the GSP auctions determines a fragility of its performance in terms of revenues, independent of the allocative distortions it may generate. The intuition behind Theorem 3 is simple, in hindsight: In the VCG mechanism, truthful bidding is dominant for the independents, irrespective of the presence of the SEMA. Hence, by manipulating the bids of its members, the agency cannot affect the bids of the independents (though it may

affect their payments, as seen in Example 3). The SEMA’s manipulation of the bids of its members therefore only has a *direct effect* on the total revenues. In the GSP auction, in contrast, the SEMA has both a *direct* and an *indirect effect* on the total revenues. The latter is due to its ability to affect the equilibrium bids of the independents.

Under the UC-restrictions, the two effects combined induce exactly the same revenue-loss as in the VCG mechanism. Since the RAE with the UC-restriction also induce efficient allocations, it may seem that Theorem 3 follows immediately from the efficiency constraint being weaker than the UC-restriction. This intuition is incorrect for two reasons. First, the UC-constraint requires the existence of *feigned valuations* which can rationalize the observed bid profile, but does not require that they preserve the ranking of the true valuations. Second, when the exogenous restrictions $\mathcal{R} = (R_{\mathcal{C}})_{\mathcal{C} \subseteq \mathcal{I}: |\mathcal{C}| \geq 2}$ are changed, they change for all coalitions: hence, even if $R_{\mathcal{C}}$ is weaker for any given \mathcal{C} , the fact that it is also weaker for the subcoalitions may make the stability constraint (S.2) more stringent. Which of the two effects dominates, in general, is unclear. Hence, because of the ‘farsightedness assumption’ embedded in constraint (S.2), the proof of the theorem is by induction on the size of the coalition.

The next example illustrates the Eff-RAE in the environment of Examples 3 and 4. Table 1 compares the bid profiles and revenues of the equilibria illustrated in our leading examples.

Example 5. *Consider the environment of Examples 3 and 4, with $\mathcal{C} = \{1, 3\}$. The efficiency-constrained RAE is $\hat{b} = (\hat{b}_1, 2.8, 1.6^+, 1.6, 1)$, which results in revenues 82, which are lower than the RAE in VCG mechanism (86). Note that, relative to the UC-RAE in Example 4, the coalition lowers b_3 to the lowest level consistent with the efficient ranking. This in turn induces independent bidder 2 to lower his bids, hence the extra revenue loss is due to further direct and indirect effects. We note that the efficiency restriction is not binding in this example, and hence the Eff-RAE and the unconstrained RAE coincide.*

Since, under the efficiency restriction, the GSP auction induces the same allocation as the VCG mechanism, the two mechanisms are ranked in terms of revenues purely due to the SEMA’s effect on prices. Obviously, if allocative inefficiencies were introduced, they would provide a further, independent source of revenue reduction. As already noted, this is not

Table 1: Summary of Results in Examples

Valuations	VCG	GSP (EOS)	RAE in VCG	UC-RAE in GSP	(Eff.) RAE in GSP
5	5	\mathbf{b}_1	\mathbf{b}_1	\mathbf{b}_1	\mathbf{b}_1
4	4	3.15	4	2.9	2.8
3	3	2.3	2^+	1.8	1.6^+
2	2	1.6	2	1.6	1.6
1	1	1	1	1	1
Revenues	96	96	86	86	82

Summary of results in examples 1, 3-5. Coalition members' bids and valuations are in bold. The VCG and GSP columns represent the competitive equilibria in the two mechanisms as described in example 1. The RAE in VCG and the revenue equivalent UC-RAE in the GSP are from examples 3 and 4 respectively. The last column denotes both the Efficient RAE and the unrestricted RAE of the GSP auction, which coincide in example 5.

the case in Example 5, in which the efficiency constraint is not binding. Unlike the VCG mechanism, however, the unrestricted RAE of the GSP auction can be inefficient as well:

Example 6. Consider a case with $n = 8$, CTRs $x = (50, 40, 30.1, 20, 10, 2, 1, 0)$ and valuations $v = (12, 10.5, 10.4, 10.3, 10.2, 10.1, 10, 1)$. Let $\mathcal{C} = \{5, 6\}$. The unrestricted RAE is essentially unique (up to the highest overall bid) and inefficient, with the coalition bidders obtaining slots 4 and 6. Equilibrium bids (rounding off to the second decimal) are $b = (b_1, 9.91, 9.76, 9.12, 9.5, 7.94, 5.5, 1)$. The inefficiency arises as follows. The agency drastically reduces the bid of its lower-valued member to benefit the other member. This, however, creates incentives for the independents $i < 5$ to move down to the position just above bidder 6, in order to appropriate some of the rents generated by the reduction of its bid. In order to prevent these independents from doing so, 5's bid must also be reduced, to make the higher positions more attractive. But in this example, the reduction of 6's bid is large enough that the undercut of 4 is low enough that the coalition actually prefers giving up slot 5 to the independent, and climb up to the higher position. Thus, the coalition does not benefit directly from the reduction of 6's bid, but indirectly, by attracting 4 to the lower position.

4.3 Discussion of the theoretical results

The VCG mechanism is typically regarded as a poor mechanism under collusion. Yet, we have shown that, in the presence of a SEMA, it outperforms the GSP auction both in

terms of revenues and allocative efficiency. On the one hand, this conclusion provides a remarkably negative result for the GSP auction, which highlights an important fragility of this mechanism with respect to the possibility of coordinated bidding. On the other hand, the efficiency result in Theorem 1 shows that the desirable allocative properties of the VCG mechanism are robust to the introduction of a SEMA, suggesting that the VCG mechanism may be more resilient than we might have expected.

Independent of the efficiency result, which we discuss below, a key source of the resilience of the VCG mechanism is its strategy-proofness: if individual bidders have a dominant strategy, then a SEMA may have at most a *direct effect* on the total revenues, as independents have no reason to adjust their bids as a function of the strategy and composition of the agency. In the GSP auction, in contrast, the SEMA can manipulate the bids of its members and induce lower bids also amongst the independents. As shown in Example 4, the resulting *indirect effect* may be significant: due to the complex equilibrium effects of the GSP auction, even a small manipulation of the coalition’s bids may have a large effect on total revenues. In general, indirect effects can be completely avoided only if independents’ best responses are unaffected by the bids of the agency. This suggests that strategy-proofness may be a desirable property in the presence of a SEMA.

The efficiency result of Theorem 1 clearly relies on the assumptions of our model. As illustrated in Example 2, without the recursive stability constraint (S.2), coordinated bidding in the VCG mechanism may induce inefficiencies. But as soon as individuals’ incentives to abandon the coalition are taken into account, efficiency is restored.¹⁹ Thus, the significance of the efficiency result depends on whether or not we believe that a mechanism’s performance should be evaluated taking into account the underlying coalition formation process. For a medium or long-run perspective, we think that this is important, hence we built a ‘farsightedness assumption’ into our model in the spirit of Ray and Vohra’s. Besides its intrinsic theoretical appeal, this approach also proved convenient from an analytical viewpoint: in addition to being key to the efficiency result in Theorem 1, that assumption was crucial to make the analysis of the GSP tractable, and deliver clear economic insights.

¹⁹Bachrach (2010) studies collusion in the VCG mechanism in a classical cooperative setting (i.e. without distinguishing the SEMA clients from the independents, and without Ray and Vohra’s farsightedness assumption), finding that the VCG is vulnerable to this form of collusion.

5 Application to Search Auctions Data

In this section we apply our model to data provided by a search engine.²⁰ The data observed by the search engine include information on all the variables entering our model, with the exception of advertisers’ valuations. In particular, the search engine records the advertisers’ identity, their SEMA (if any), bids, positions and clicks received. These data allow it to estimate CTRs as click frequencies. Within a set of “historical” data preserved for research purposes, this search engine identified all those keywords where one single SEMA submitted bids for two advertisers. Then, given these keywords, it constructed a sample with all the auctions in which the SEMA places bids for both advertisers. These auctions were held during a randomly selected set of days within a three-month time window between 2010 and 2011. The resulting sample has 71 keywords.²¹ These keywords are from different industries and belong to a set of keywords that are rather frequently searched. Different SEMAs and advertisers appear across the keywords, but, for any given keyword, the set of advertisers winning slots is fairly stable.²²

A major feature of these data is that they also record ‘quality scores’. In the variant of the GSP auction used by this search engine, these quality scores are the advertisers’ idiosyncratic score assigned by the search engine to account for various quality dimensions, including the CTRs. Quality scores concur in determining the assignment of advertisers to slots and prices: advertisers are ranked by the product of their bid and quality score, and pay a price equal to the minimum bid consistent with keeping that position.

Formally, letting e_i denote the score of bidder i , advertisers are ranked by $e_i \cdot b_i$, and CTRs are equal to $e_i \cdot x^{\rho(i)}$, the product of a ‘quality effect’ and a ‘position effect’. The price paid by bidder i in position $\rho(i)$ is $p_i = e^{\rho(i+1)} b^{\rho(i+1)} / e_i$.²³ Relabeling advertisers so that

²⁰By agreement with the data provider, we cannot reveal its name.

²¹The data allows us to unambiguously distinguish advertisers from SEMAs. Among advertisers, however, different accounts are held by different units of the same firm or by different firms that are linked via common ownership. An ad hoc procedure was performed to eliminate from the sample all such ambiguous cases.

²²While popular keywords typically attract many advertisers, the fact that for each auction retained in the dataset the pair of agency advertisers must bid implies greater homogeneity across auction participants.

²³In extending the model to accommodate quality scores, we again follow EOS and Varian (2007). Clearly, the baseline model of the previous sections obtains letting $e_i = 1$ for all i .

$e_i v_i > e_{i+1} v_{i+1}$, the competitive (EOS) equilibrium bids are such that, for all $i = 2, \dots, S$,

$$e_i v_i = \frac{e_i b_i x^{i-1} - e_{i+1} b_{i+1} x^i}{x^{i-1} - x^i} \geq \frac{e_{i+1} b_{i+1} x^i - e_{i+2} b_{i+2} x^{i+1}}{x^i - x^{i+1}} = e_{i+1} v_{i+1}. \quad (10)$$

This is the analogue, with quality scores, of what is implied by equation (2) in Lemma 1. As shown below, similar modifications apply to the equilibria with agency coordination.

The rest of the section describes how to use data to address two questions. The first is how to determine whether the observed bids are more likely to result from coordinated agency behavior or from competitive bidding. The second is to quantify the revenue distortions due to agency coordination. For each question, we begin by describing the strategy used to address it and then present the results based on the data from the search engine.

5.1 Detecting Coordination

Strategy: The first use of the data that we consider regards detecting agency coordination. We propose a simple criterion that allows us to say whether the data are more likely to be generated by competitive (EOS) bidding or by one of the models of agency coordination (UC-RAE, Eff-RAE and RAE). The latter models differ from EOS in that the bids of all agency bidders, with the exception of the highest value coalition member, are ‘too low’.

This property of the coordination models leads to the following classification criterion. Consider the case of a 2-bidder coalition, the size of all coalitions in the data, and let j denote the lowest-valued agency bidder. An EOS-compatible bid profile requires that:

$$\frac{e_j b_j x^{j-1} - e_{j+1} b_{j+1} x^j}{x^{j-1} - x^j} \geq \frac{e_{j+1} b_{j+1} x^j - e_{j+2} b_{j+2} x^{j+1}}{x^j - x^{j+1}}.$$

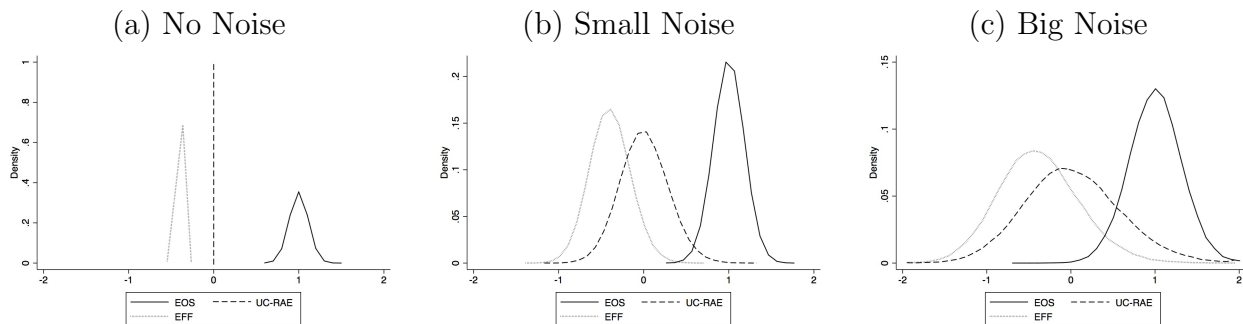
Under our coordination models, bidder j will lower its bid relative to EOS so that the above inequality either holds with equality (under UC-RAE) or is violated (under Eff-RAE or RAE). For all other bidders $i \neq j$, our coordination models do not imply any observable difference relative to EOS. Thus, a criterion based on the above inequality for bidder j captures the *only* feature which differentiates coordinated from competitive EOS bidding in 2-bidder coalitions.

Thus, given a set of T auctions, $t = 1, 2, \dots, T$ for which we observe quality scores, bids, CTRs and positions for all bidders, we can construct a quantity which captures the realization of the above inequality in each auction t :

$$J_t = \frac{e_j b_j x^{j-1} - e_{j+1} b_{j+1} x^j}{x^{j-1} - x^j} - \frac{e_{j+1} b_{j+1} x^j - e_{j+2} b_{j+2} x^{j+1}}{x^j - x^{j+1}}.$$

The distribution of J_t is potentially very informative about the types of behavior generating the data. To illustrate this point, consider panel (a) in Figure 1. It reports the distribution of J_t obtained by simulating 100,000 auctions for three equilibrium models: EOS (solid line), UC-RAE (dashed line) and Eff-RAE (dotted line). For this simulation, we fix the valuations, CTRs and coalition structure as in the example reported in Table 1, and for each auction we draw quality scores of each bidder from a Normal distribution with mean 1 and s.d. 0.03.

Figure 1: Simulation



J_t is calculated only for the low-value agency member (i.e. the bidder with $v=3$ in the example). We see that J_t is never negative when we simulate EOS, it always equals zero when we simulate UC-RAE and it is never positive when we simulate Eff-RAE.²⁴

Under the ideal conditions of the simulation, J_t allows us to unambiguously separate the bidding models. Clearly, with real data, this tool should be expected to face some limits. For instance, search engines update quality scores in real time. Hence, even if bidders can

²⁴Detecting bids as coming from UC-RAE, in which coordinated bids were defined as ‘undetectable’, may strike as oxymoronic. The reason is that UC-RAE is undetectable in a single auction, but because it entails that J_t is exactly zero, it becomes detectable once many auctions are considered: $J_t = 0$ in every auction would be possible only if valuations were changing with the quality scores in an ad hoc way, hence the detectability of UC-RAE across auctions.

frequently readjust bids, it is not the case that bids are always optimized for the ‘true’ quality scores. The model of incomplete information developed by Athey and Nekipelov (2012) to address this limit of EOS, however, reveals that the departures from EOS are small. This is also consistent with the findings in Varian (2007) showing that small perturbations in beliefs about quality scores are sufficient to reconcile observed data with EOS bidding.²⁵

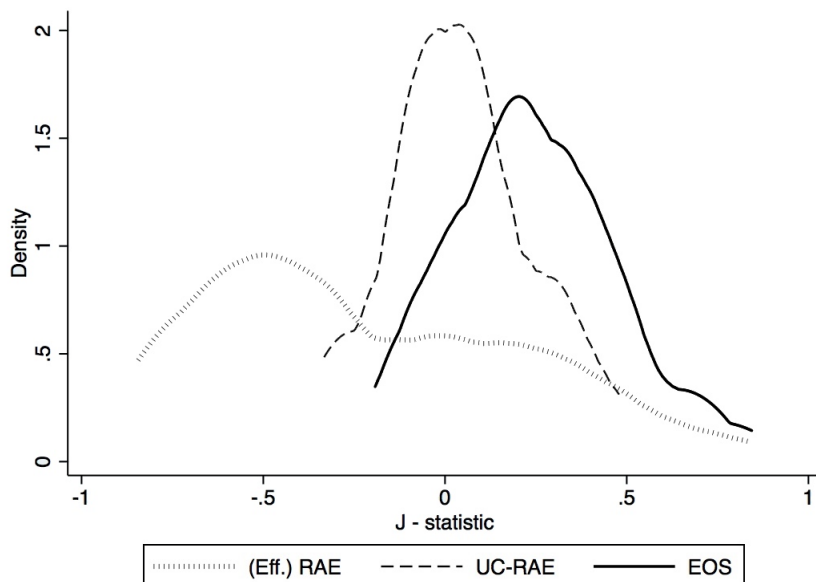
Albeit small, the presence of belief errors about quality scores can impact J_t . To illustrate this point, in plot (b) and (c) of Figure 1 we repeat the previous simulation under two scenarios. In both cases, we consider a belief error that enters multiplicatively: for each bidder i and auction t the true quality score is e_{it} , but bidders believe the true score to be \tilde{e}_{it} , where $\tilde{e}_{it} = d_{it} \cdot e_{it}$. Panel (b) considers the case of a small error, with $d_{it} \sim \mathcal{N}(1, 0.05^2)$, while panel (c) considers the case of a larger error, with $d_{it} \sim \mathcal{N}(1, 0.1^2)$. These two cases illustrate that, with any belief error, the distribution of J_t under UC-RAE is no longer degenerate at zero. This implies the need to search for UC-RAE cases by looking at an interval around zero, thus introducing some arbitrariness in the use of the J_t criterion. Moreover, overlaps in the three distributions make it more ambiguous to discriminate between the different models. In panel (b), the relatively small amount of noise still allows us to correctly classify the bidding models by looking at whether most of the mass of the distribution lies to the left of zero, around zero or to the right of zero. In practice, this can be operationalized in many ways by looking, for instance, at the smallest interval including 80% of the mass, or, alternatively, by looking at some summary measure like mean, median or mode. As shown by panel (c), however, when the amount of noise is big, none of these methods will give rise to a satisfactory classification. Based on the results from Athey and Nekipelov (2012) and Varian (2007), it is reasonable to expect that the case of panel (b) is the relevant one.

Results: The first set of results that we report regards detecting coordination. Thus, using the ranking and scoring rule applied at the time of the data sampling, for each key-

²⁵Varian (2007) proposed an empirical strategy to deal with small departures from EOS equilibrium strategies: namely, he assumes that advertisers play EOS strategies, but using incorrect beliefs on the quality scores: that is, bidders bid as if i ’s quality score is $d_i \cdot e_i$ instead of e_i , where the distance of d_i from 1 captures the extent of the belief error. Alternative approaches to structurally estimate the GSP auction have been proposed by Yang, Lu and Lu (2014) and Heish, Shum and Yang (2014).

word we select the lowest-placed agency bidder and then calculate the distribution of J_t for this bidder-keyword pair using all the available auctions. Figure 2 shows the distributions obtained for three different keywords. These keywords are chosen to be representative of the different types of behavior present in the data. The solid distribution is located mostly to the right of zero, thus supporting the case for EOS. The dotted distribution lies mostly to the left of zero, supporting Eff-RAE or unrestricted RAE. Finally, the dashed distribution is concentrated around zero, supporting UC-RAE.

Figure 2: Three Example Keywords



Distribution of J_t for three keywords exemplifying the different equilibria.

In addition to the caveat about belief errors explored with the simulation, the use of the J_t criterion with true data poses an additional concern. The simulations in Figure 1 use data from i.i.d. draws, but real auctions can be correlated in many ways. Understanding the linkages between auctions is a complex problem (Choi and Varian, 2012) and addressing it goes well beyond the scope of this paper. Thus, we still use the J_t distributions obtained from the data to classify the keywords, but warn that this classification should be taken with a grain of salt. Under this caveat, the results of our classification show that 36 keywords are compatible with UC-RAE, 32 with EOS and 3 with (Eff-)RAE.²⁶ In the following, we will

²⁶The classification is obtained by looking at whether a 95 percent confidence interval for the median of

use the 36 keywords classified as UC-RAE to quantify the revenue losses due to coordinated bidding.

5.2 Revenue Effects

Strategy: The second use of the data that we consider consists of evaluating the revenue effects of coordination. Under our coordination models, there exists a one-to-one mapping between the independents' bids and their valuations. We can thus invert their bids to recover valuations, and use them to construct upper and lower bounds for the agency bidders. To see this, suppose that we observe a 2-bidder coalition that bids according to one of our models of coordination. Then, if j is the lowest valued agency member, his value is bounded below by the value of the bidder in position $\rho(j + 1)$ and above by the bidder in position $\rho(j - 1)$ (or by the bidder in position $\rho(j - 2)$, if the two agency bidders are contiguous). Although no bound can be derived when the coalition occupies the top two slots or when its lowest valued member has no bidder below it, in all other cases this approach is informative and allows us to construct counterfactual revenues under competitive bidding.

To reconcile this approach with the belief errors discussed above, when inverting bids into valuations, we follow Varian (2007) and assume that the realized belief errors are the smallest errors required to rationalize the data as coming from equilibrium bidding. More specifically, focusing on the instances identified as UC-RAE – which constitute the majority of keywords in our data and also provide a lower bound to the revenue effects of bid coordination – define $\tilde{b}_i = d_i e_i b_i$. The program that we solve to find the smallest d under the UC-RAE restrictions is:

$$\begin{aligned} \min_d \sum_{i>1} (d_i - 1)^2 \quad \text{subject to:} \\ \left\{ \begin{array}{ll} \frac{\tilde{b}_i x^{i-1} - \tilde{b}_{i+1} x^i}{x^{i-1} - x^i} \geq \frac{\tilde{b}_{i+1} x^i - \tilde{b}_{i+2} x^{i+1}}{x^i - x^{i+1}}, & \text{if } i \in I \setminus \mathcal{C} \text{ or } i \in \{\min(\mathcal{C})\}; \\ \tilde{b}_i x^{i-1} = \frac{x^{i-1} - x^i}{x^{i+1} - x^{i+2}} [\tilde{b}_{i+2} x^{i+1} - \tilde{b}_{i+3} x^{i+2}] + \gamma d_i e_i [x^{i-1} - x^i] + \tilde{b}_{i+1} x^i, & \text{if } i \in \mathcal{C} \setminus \{\min(\mathcal{C})\}; \end{array} \right. \end{aligned}$$

J_t lies to the left of zero, or includes zero, or lies to the right of zero. Similar results are obtained with the alternative methods described earlier involving the mode or the concentration of the mass of the distribution. For the mean, instead, the presence of outliers produces different and less reliable results.

where γ is the minimum bid increment (5 cents in the data). For the highest valued agency member, we treat it as an independent bidder, since its bid is not unique under the UC-RAE.

Results: Our second set of results concerns the effects of bid coordination on revenues. We focus on the 36 keywords classified as compatible with UC-RAE. Using the procedure described earlier, we obtain upper and lower bounds for the value of the lowest placed coalition member. We then compute counterfactual bids assuming competitive EOS bidding under the two scenarios: when the value of the lowest placed agency bidder equals its lower bound and when it equals its upper bound. This procedure is applied to each individual auction.

Separately for each keyword, we calculate the (expected) revenues by multiplying bids with CTRs and summing across all bidders and all auctions. We end up with three variables - observed, lower bound and upper bound revenues - each with 36 realizations. In Table 2, we compare the means of these variables reporting 95 percent confidence intervals for a t-test that the mean difference equals zero. This is a test for matched data where the keyword is the matching variable. Once again, we warn that the potential correlation across auctions in the data requires caution in interpreting the findings.

Table 2: Revenue Effects for the 36 UC-RAE Keywords

	Observed	Lower Bound	$\Delta=Obs.-LowerB.$	Upper Bound	$\Delta=UpperB.-Obs.$
Agency	33.20	32.14	1.06 [0.69; 1.42]	35.28	2.08 [1.49; 2.68]
Others	66.80	64.16	2.64 [1.70; 3.58]	72.62	5.82 [3.73; 7.91]
Total	100	96.30	3.7 [2.47; 4.92]	107.90	7.9 [5.32; 10.44]

For each of the 36 keywords, revenues are expressed as a percentage of the total observed revenues. The table reports the mean revenues across the 36 keywords for both observed and counterfactual (lower and upper bound) cases. Δ indicates the differences in the means. The values in the squared bracket are the endpoints of a 95% confidence interval for matched differences in means.

Nevertheless, the results of this analysis are interesting. The table columns report the mean revenues for observed, lower bound and upper bound revenues, as well as the mean

differences. The table rows separate the total revenues (reported in the last row), into the revenues generated by agency bidders (top row) and by the other bidders (middle row). All revenues are expressed as a percentage of the total observed revenues (bottom left corner of the table).

The findings help to understand both the size of the potential effects of coordination, as well as the division between direct and indirect effects. The confidence intervals (reported in squared brackets) indicate that the differences between observed and counterfactual revenues are statistically significant for both upper and lower bound mean revenues. In terms of magnitude, however, observed revenues are closer to the lower than to the upper bound, as expected when the UC-RAE fits the data.

Furthermore, our upper bound estimates point to a potential revenue loss of 7.9 percent, which is an economically large effect given the growing role of SEMAs in the multi billion dollar revenues generated by the GSP. These findings also reveal that the indirect effect is quantitatively important: only one quarter of the estimated 7.9 percent loss comes from the direct effect of agency bids, while the remaining three quarters are due to the indirect effect.

6 Conclusions

This is the first study to focus on the role of agencies on sponsored search auctions, and in particular on their role in coordinating the bids of different advertisers. Our theoretical results uncover a striking fragility of the GSP auction to bid coordination, and our empirical analysis provides evidence that even the small 2-bidder coalitions frequently observed in the data can have large effects on revenues. Given that the GSP auction is used by all major search engines to sell advertisement space worth billions of dollars, our findings show that the rapid diffusion of SEMAs in this market has potentially large economic consequences.

From a methodological perspective, we note that the notion of RAE – and particularly the *‘farsightedness’* idea – has been key to obtain clear results in this complicated auction, in which competitive and coordinated bidding coexist. This suggests that this broader approach, which combines cooperative and non-cooperative ideas, may be fruitful to address the important problem of partial cartels, an outstanding challenge in the literature.

Clearly, our results are also interesting from a market design perspective. While beyond the scope of this paper, our analysis suggests some possible guidelines for research in this area. For instance, our analysis of the GSP auction with ‘undetectable coordination’ constraints implicitly suggests a way of deriving reservation prices to limit the impact of bids coordination. This kind of intervention would thus reinforce the resilience of the GSP auction, without entailing major changes in the mechanism. More radical modifications of the mechanism may be pursued as well. Theorem 1 shows that, in this setting, the VCG mechanism performs surprisingly well in the presence of bid coordination. As discussed in section 4, this is largely due to the strategy-proofness of this mechanism. While the complexity of the VCG payments is often seen as an impediment to the actual implementability of this mechanism, our analysis suggests that strategy-proofness may be a desirable property for a mechanism to perform well in the presence of bid coordination. Thus, variations of uniform price auctions may also be simpler and more viable options to address bid coordination.

Finally, as pointed out earlier, our findings are also potentially relevant from an antitrust perspective. In particular, the agency behavior in our model is analogous to that of buying consortia, which have been sanctioned in the past (see footnote 5). Nevertheless, the specificities of the sponsored search advertisements market suggest a more nuanced view of the harm to the consumers. First, although multiple search engines exist, the degree of competition between them is likely substantially less than that between most of the advertisers. Since the lower auction prices imply a reduction in the marginal cost advertisers pay to reach consumers, advertiser competition would thus imply that some savings are passed on to consumers. Therefore, harm to consumers would result only if the agency engages in coordinating not only the auction bids, but also the prices charged to consumers. Second, bid coordination can negatively affect the quality of the service received by consumers by exacerbating further the advantage of dominant search engines relative to fringe ones. In Europe, for instance, where 90% of the searches pass via Google, agencies might be rather careful not to harm Google given the risk of being excluded from its results page. Smaller search engines cannot exert such a threat because agencies are essential to attract new customers. The shift of revenues from small search engines to marketing agencies could thus deprive the former of the essential resources needed for technology investments. Thus, to

the extent that competing search engines exert pressure for quality improvements, bid coordination poses a potential threat to consumer welfare.²⁷ All these considerations represent potentially fruitful directions for future research.

A Appendix

A.1 Technical Details

As discussed in section 2 (p. 6), any generic profile $b_{-i} = (b_j)_{j \neq i}$ partitions the space of i 's bids, \mathbb{R}_+ , into $S + 1$ intervals: $[0, b_{-i}^S)$, $[b_{-i}^S, b_{-i}^{S-1})$, \dots , $[b_{-i}^1, \infty)$. Letting $b_{-i}^0 \equiv \infty$ and $b_{-i}^{S+1} \equiv 0$, if bidder i bids $b_i \in (b_{-i}^t, b_{-i}^{t-1})$, then he obtains slot $t = 1, \dots, S + 1$ at per-click-price b^t . If b_i is placed at one extreme of such intervals, the allocation is determined by the tie-breaking rule embedded in the function ρ . The function π_i introduced in section 2 can be seen as a correspondence $\pi_i : \mathbb{R}_+^{n-1} \rightrightarrows \{1, \dots, S + 1\}$ such that for each $b_{-i} \in \mathbb{R}_+^{n-1}$, $\pi_i(b_{-i}) = \arg \max_{t=1, \dots, S+1} (v_i - b_{-i}^t) x^t$.²⁸

To allow for the possibility of ties in the bids profiles, it is necessary to generalize some of these concepts. In particular, if some of i 's opponents place equal bids (i.e., $b_{-i} = (b_j)_{j \neq i}$ is such that $b_j = b_k$ for some $j \neq k$), then, depending on the tie-breaking rule embedded in ρ , some of the $S + 1$ positions may be precluded to player i (e.g., if $i = 1$, and $b_2 = b_3$, if the tie-breaking rule is specified as in footnote 6, position $s = 2$ is precluded to player i). In that case, the argmax in the definition of π_i should be taken over the set of positions that are actually accessible to i . Formally: for any $b_{-i} \in \mathbb{R}_+^{n-1}$, let

$$\mathcal{S}(b_{-i}) = \{s = 1, \dots, S + 1 : \exists b_i \text{ s.t. } \rho(i; b_i, b_{-i}) = s\}.$$

²⁷Quality of the links is indeed considered relevant for antitrust actions. For instance, one of the claims in the ongoing Google case before the European antitrust authority is the alleged abuse by Google of its dominant position to present links of inferior quality by directing consumers to Google's own outlets.

²⁸This correspondence is always non-empty valued, and multi-valued only if i is indifferent between two positions. We can ignore this case here (for instance, assuming that such ties are always broken in favor of the lower position) and treat $\pi_i : \mathbb{R}_+^{n-1} \rightarrow \Pi$ as a function (if not, π_i should be thought of as a selection from the correspondence above).

Then, we redefine the function $\pi_i : \mathbb{R}_+^{n-1} \rightarrow \{1, \dots, S+1\}$ as follows: for every $b_{-i} \in \mathbb{R}_+^{n-1}$

$$\pi_i(b_{-i}) \in \arg \max_{s \in \mathcal{S}(b_{-i})} (v_i - b_i^s) x^s.$$

Since $\mathcal{S}(b_{-i})$ is always non-empty and finite, the best responses $BR_i : \mathbb{R}_+^{n-1} \rightrightarrows \mathbb{R}_+$ defined in Section 2 (p.6) is well-defined, and so is $BR_i^* : \mathbb{R}_+^{n-1} \rightrightarrows \mathbb{R}_+$ in (1). With these changes to the definition of π_i , the rest of the analysis also extends to the case of ties in bids.

A.2 Proofs

All the results are proven for the case in which $n = S+1$. The extension to the general case is straightforward but requires more cumbersome notation.

A.2.1 Proof of Lemma 1

Let $b \in \mathcal{EG}(v)$. By definition, for any i , $\rho(i) = s$ implies $\pi_i(b_{-i}) = s$ if $s \leq S$ and $\pi_i(b_{-i}) = S+1$ if $s > S$. Hence, $b_{-i}^{\pi_i(b_{-i})} = b^{s+1}$ whenever $s \leq S$. Now, for any i such that $\rho(i) \leq S$ and j s.t. $\rho(j) = \rho(i) + 1$, the following must hold:

$$\text{by the optimality of } b_i : \quad (v_i - b^{\rho(i)+1}) x^{\rho(i)} \geq (v_i - b^{\rho(i)+2}) x^{\rho(i)+1}; \quad (11)$$

$$\text{by the condition in (1) for } j : \quad (v_j - b^{\rho(i)+2}) x^{\rho(i)+1} = (v_j - b^{\rho(i)+1}) x^{\rho(i)}. \quad (12)$$

Rearranging, we obtain

$$v_i (x^{\rho(i)} - x^{\rho(i)+1}) \geq b^{\rho(i)+1} x^{\rho(i)} - b^{\rho(i)+2} x^{\rho(i)+1} = v_j (x^{\rho(i)} - x^{\rho(i)+1}),$$

which implies that $v_i > v_j$ (since, by assumption, $(x^s - x^{s+1}) > 0$ for any $s \leq S$ and $v_i \neq v_j$ for any $i \neq j$). Hence, all agents who placed the $S+1$ bids in an equilibrium are ranked efficiently among themselves. For the others, notice that for any i such that $\rho(i) > S$, the indifference condition in the definition of BR_i^* (eq. 1) requires that $0 = (v_i - b_i) x^S$, hence $v_i = b_i$ whenever $\rho(i) > S$. Hence, agents not obtaining a position bid their valuation. It follows that $b^i = b_i$ for all i (agents with higher valuations bid higher) and $b_i = v_i$ for

all $i \geq S + 1$. Equation (2) follows immediately, applying the indifference condition (eq. 1) for agents $i = 2, \dots, S$ with the initial condition $b_{S+1} = v_{S+1}$. Since this recursion does not pin down b_1 , the only restriction imposed on that is that $b_1 > b_2$. Finally, note that the recursion in (2) coincides with EOS' LREF equilibrium (EOS, Theorem 2), and with Varian's lower-bound symmetric Nash Equilibrium (Varian, 2007, eq.9).

A.2.2 Proof of Theorem 1

We prove the statement by induction on the size of the coalition.

The **induction basis** is the non-collusive benchmark (or, equivalently, $|\mathcal{C}| = 1$). In this case all players use their dominant strategies, $b_i = v_i$ for each i , which clearly ensures $v_i \in (b_{i+1}, v_{i-1})$ for all i , and the equilibrium bids profile is as claimed in the theorem (in fact, a stronger statement holds, as even the bid of the highest coalition member is exactly pinned down).

We now proceed to the **inductive step**. Suppose we have shown that the result holds for all coalitions \mathcal{C}' such that $|\mathcal{C}'| < C$. We want to show that it also holds for \mathcal{C} such that $|\mathcal{C}| = C$. Let i be the member who is placing the lowest bid in the coalition, and let r denote the position he occupies. Then, his payoff is equal to:

$$u_i = v_i x^r - \sum_{t=r+1}^{S+1} b^t (x^{t-1} - x^t).$$

It is useful to introduce notation to rank independent among themselves, based on their valuation. Let $v_{I \setminus \mathcal{C}} = (v_j)_{j \in I \setminus \mathcal{C}}$ and let $v_{I \setminus \mathcal{C}}(k) = v_{I \setminus \mathcal{C}}^{|\mathcal{C}|+1-k}$ denote the valuation of the k -th lowest value independent: for $k = 1$, $v_{I \setminus \mathcal{C}}(1) = v_{I \setminus \mathcal{C}}^{|\mathcal{C}|}$ is the lowest valuation among the independents, $v_{I \setminus \mathcal{C}}(2) = v_{I \setminus \mathcal{C}}^{|\mathcal{C}|-1}$ is the second lowest valuation among the independents, and so on. Now, if i is the lowest-bidding member of the coalition, all players placing lower bids are independents, and therefore bid according to their dominant strategy, $b_j = v_j$. This in turn implies that bids in positions $t = r + 1, \dots, S + 1$ are ranked efficiently between themselves, but it does not guarantee that $b^t = v_t$ for each $t \geq r + 1$, unless all $j \in \mathcal{C}$ are such that $j \leq r$. Thus, we can only conclude that bids b^t for $t = r + 1, \dots, S + 1$ are placed

by the lowest-valued $S + 1 - r$ independents. It follows that we can rewrite u_i as follows:

$$u_i = v_i x^r - \sum_{t=r+1}^{S+1} v_{I \setminus \mathcal{C}}(S + 2 - t) (x^{t-1} - x^t). \quad (13)$$

Let us consider the function $\tilde{u}_i(r)$ of player i 's payoff as a function of the position r occupied by player i , given that he is the lowest-bidder in the coalition. Define $u_i^* = \max_r \tilde{u}_i(r)$. Clearly, u_i^* provides an upper bound to the payoff that i can achieve as a lowest bidding member of the coalition. In the following, we will show that if $i \neq \max\{j : j \in \mathcal{C}\}$, such upper bound u_i^* is less than the payoff $u_i^{\mathcal{C} \setminus \{i\}}$ that i would obtain by leaving the coalition. Hence, the coalition is stable only if the lowest bidding member is also the member with the lowest valuation.

Let $\mathcal{J} = \{j \in \mathcal{C} : j > i\}$ be the set of coalition members with values lower than v_i . We will show that

$$u_i^* = v_i x^{i+|\mathcal{J}|} - \sum_{t=i+|\mathcal{J}|+1}^{S+1} v_{I \setminus \mathcal{C}}(S + 2 - t) (x^{t-1} - x^t). \quad (14)$$

First we show that \tilde{u}_i is maximized only if i is placed efficiently with respect to the independents. That is, for any $j \in I \setminus \mathcal{C}$, $j < i$ if and only if $\rho(j) < r$. We proceed by contradiction: suppose that there exist $j \in I \setminus \mathcal{C}$ such that either $j < i$ and $\rho(j) > r$, or $j > i$ and $\rho(j) < r$.

Consider the first case: Since independents are ranked efficiently among themselves, for any $j, l \in I \setminus \mathcal{C}$, $l < j$ if and only if $\rho(l) < \rho(j)$. It follows that if there exists $j \in I \setminus \mathcal{C} : j < i$ and $\rho(j) > r$, such j can be chosen so that $j = r + 1$, i.e. j occupies the position immediately following i 's. We next show that, in this case, u_i would increase dropping down one position, i.e. switching from position r to $r + 1$. To see this, consider u_i as a function of the position r occupied by player i , $\tilde{u}_i(r)$, and notice that

$$\begin{aligned} \tilde{u}_i(r + 1) - \tilde{u}_i(r) &= v_i (x^{r+1} - x^r) + v_{I \setminus \mathcal{C}}(S + 1 - r) (x^r - x^{r+1}) \\ &= (v_{I \setminus \mathcal{C}}(S + 1 - r) - v_i) (x^r - x^{r+1}), \end{aligned}$$

where v_{r+1} is the valuation of the highest independent if i occupies position r . Since, by

assumption, $x^r > x^{r+1}$, it follows that

$$\text{sign}(\tilde{u}_i(r+1) - \tilde{u}_i(r)) = \text{sign}(v_{I \setminus \mathcal{C}}(S+1-r) - v_i).$$

Under the absurd hypothesis, $v_{I \setminus \mathcal{C}}(S+1-r) > v_i$, hence u_i increases dropping one position down. A similar argument shows that in the second case of the absurd hypothesis, i.e. if there exists $j \in I \setminus \mathcal{C} : j > i$ and $\rho(j) < \rho(i)$, u_i could be increased climbing one position up, from r to $(r-1)$. The result obtains considering the difference

$$u_i(r) - u_i(r-1) = (v_{I \setminus \mathcal{C}}(S+2-r) - v_i)(x^{r-1} - x^r),$$

which is negative because $v_{I \setminus \mathcal{C}}(S+2-r) < v_i$ under the absurd hypothesis.

We have thus proved that u_i is maximized at position r such that for any $j \in I \setminus \mathcal{C}$, $j < i$ if and only if $\rho(j) < r$.

Hence, in the arrangement that maximizes the utility of the lowest bidder in the coalition, this bidder is placed efficiently with respect to the independents, and only independents are below him. The lowest coalition bidder i therefore occupies position $i + |\mathcal{J}|$. That is, he scales down from his efficient position exactly a number of positions equal to the number of members of the coalition with lower valuations that are placed above him. (Clearly, i occupies the i -th position if and only if $\mathcal{J} = \emptyset$, i.e. if i , the lowest bidding member of the coalition, also has the lowest value in the coalition. Equation (14) follows from (13) setting $r = i + |\mathcal{J}|$.)

The next step is to show that when $\mathcal{J} \neq \emptyset$, $u_i^* < u_i^{\mathcal{C} \setminus \{i\}}$, (i 's payoff if he abandons the coalition). To this end, assume $\mathcal{J} \neq \emptyset$ and let \bar{b}_k be the bid of bidder k in the equilibrium with coalition $\mathcal{C} \setminus \{i\}$. Since, under the inductive hypothesis, the equilibrium with coalition $\mathcal{C} \setminus \{i\}$ is efficient, $\bar{b}_k = \bar{b}^k$ for any k , and from (??) we obtain:

$$u_i^{\mathcal{C} \setminus \{i\}} = v_i x^i - \sum_{k=i+1}^{S+1} \bar{b}_k (x^{k-1} - x^k).$$

By the inductive hypothesis, the equilibrium with this smaller coalition is as in the

Theorem's statement. Thus, in particular, we have that $\bar{b}_k < v_{k-1}$ for all $k \in I$ (if k is an independent, because he bids $\bar{b}_k = v_k < v_{k-1}$; if he's the highest-value member of the coalition, because $\bar{b}_k \in (b_{k+1}^+, v_{k-1})$, otherwise $\bar{b}_k = b_{k+1}^+ < v_{k-1}$). We also show that $\bar{b}_k \leq v_{I \setminus C}(S+2-k)$ for all k . To this end, observe that all $k \geq \max\{\mathcal{J}\}$ are independents (both before and after i drops out), so that for all $k \geq \max\{\mathcal{J}\}$, $\bar{b}_k = v_k = v_{I \setminus C}(S+2-k)$: these are the lowest bidding and the lowest-value bidders, hence also the lowest independents. For $k < \max\{\mathcal{J}\}$, at least one of the $S+2-k$ elements of the set $\{k, k+1, \dots, S+1\}$ is a member of the coalition. It follows that the valuation of the $(S+2-k)$ -th lowest independent is higher than v_k , hence $v_{I \setminus C}(S+2-k) \geq v_{k-1}$, which in turn implies $v_{I \setminus C}(S+2-k) > \bar{b}_k$. Overall, we have that $\bar{b}_k < v_{k-1}$ and $\bar{b}_k \leq v_{I \setminus C}(S+2-k)$ for all $k \in I$. Using the first inequality for $k \leq i + |\mathcal{J}|$ and the second inequality otherwise, we see that if $\mathcal{J} \neq \emptyset$,

$$\begin{aligned} u_i^{C \setminus \{i\}} &= v_i x^i - \sum_{k=i+1}^{i+|\mathcal{J}|} \bar{b}_k (x^{k-1} - x^k) - \sum_{k=i+|\mathcal{J}|+1}^{S+1} \bar{b}_k (x^{k-1} - x^k) \\ &> v_i x^i - \sum_{k=i+1}^{i+|\mathcal{J}|} v_{k-1} (x^{k-1} - x^k) - \sum_{k=i+|\mathcal{J}|+1}^{S+1} v_{I \setminus C}(S+2-k) (x^{k-1} - x^k) \quad (15) \end{aligned}$$

Combining (14) and (15), we get

$$\begin{aligned} u_i^{C \setminus \{i\}} - u_i^* &> v_i (x^i - x^{i+|\mathcal{J}|}) - \sum_{k=i+1}^{i+|\mathcal{J}|} v_{k-1} (x^{k-1} - x^k) \\ &\geq v_i (x^i - x^{i+|\mathcal{J}|}) - v_i (x^i - x^{i+|\mathcal{J}|}) = 0, \end{aligned}$$

where the latter inequality follows because $v_{k-1} \leq v_i$ for all $k \geq i+1$. Hence, whenever $\mathcal{J} \neq \emptyset$, we obtain $u_i < u_i^{C \setminus \{i\}}$: that is, the recursive stability condition (S.2) is violated for bidder i . $\mathcal{J} = \emptyset$ therefore is a necessary condition for equilibrium. Hence, in any equilibrium, the lowest-bidder in the coalition also has the lowest valuation in the coalition. Moreover, if $\mathcal{J} = \emptyset$, $u_i^* = u_i^{C \setminus \{i\}}$ (by equations (14) and (15)), hence in equilibrium $u_i = u_i^*$ and $i = \rho(i)$:

$$u_i = v_i x^i - \sum_{k=i+1}^{S+1} v_k (x^{k-1} - x^k) = u_i^{C \setminus \{i\}}. \quad (16)$$

Furthermore, since the payment of coalition members above i is strictly decreasing in b_i and (subject to the constraint that $b_i \in (b_{i+1}, b_{i-1})$) positions are independent of b_i , we know that the coalition will set b_i as low as possible to ensure i 's efficient position. That is, $b_i = b_{i+1}^+ = v_{i+1}^+$.

We have determined the positions and bids of all bidders $k \geq i$. We know that the remaining coalition members are positioned above these bidders and do not affect u_i . Thus, the remaining task for the coalition is to choose bids $(b_j)_{j \in \mathcal{C} \setminus \{i\}}$ in order to maximize $\sum_{j \in \mathcal{C} \setminus \{i\}} u_j$, subject to the constraint that $b_j > b_i$ for all $j \in \mathcal{C} \setminus \{i\}$. We now need to look separately at two cases: $|\mathcal{C}| = 2$ and $|\mathcal{C}| > 2$.

First, if $|\mathcal{C}| = 2$, the task is simply to maximize the payoff of the other member of the coalition, j , by determining his position relative to the remaining independents. But this, by the usual argument, is achieved when j is placed efficiently with respect to these independents. This is achieved if and only if $b_j \in (b_{j+1}, v_{j-1})$.

Second, if $|\mathcal{C}| > 2$, note that even when one of the members $j \in \mathcal{C} \setminus \{i\}$ drops out, i still remains a non-top member of the coalition. Hence, its bid does not change. Naturally, the bids of all $k > i$ (who are independents) do not change either. Hence, the payoffs of all bidders $k < i$ both before and after one of the coalition members (other than i) drops out are shifted by the same constant relative to a game in which the bidders $k \geq i$ (and the corresponding slots) are removed: thus, the presence of these bidders has no effect on either the payoffs or the outside options. It follows that the problem we are solving at this stage is exactly equivalent to finding the equilibrium in the VCG game played between coalition $\mathcal{C} \setminus \{i\}$ and independents $\{j \in I \setminus \mathcal{C} : j > i\}$ with slots x^1, \dots, x^{i-1} . This game has coalition size $C - 1$, so the solution follows by the inductive hypothesis. ■

A.2.3 Proof of Theorem 2

Since the UC-restrictions imply the stability restriction (S.1), the agency's problem in the GSP auction with the feigned values restriction reduces to:

$$\begin{aligned}
& \max_{b_C} u_C(b_C, \beta^*(b_C)) \\
\text{subject to : } & \text{(R)} \exists v'_C \in \mathbb{R}_+^{|\mathcal{C}|}, b_{-C} \in \mathbb{R}_+^{n-|\mathcal{C}|} \text{ s.t. } (b_C, b_{-C}) \in \mathcal{EG}(v'_C, v_{-C}) \\
& \text{(S.2)} \forall i \in \mathcal{C}, u_i(b_C, \beta^*(b_C)) \geq \bar{u}_i^{\mathcal{C} \setminus \{i\}}.
\end{aligned}$$

where the equilibrium conjectures β^* are such that, $\forall b_C, \beta^*(b_C) \in \mathbb{R}_+^{n-|\mathcal{C}|}$: $\forall i \in I \setminus \mathcal{C}, b_i^* \in BR_i^*(b_C, b_{-i, -C}^*)$. The proof exploits the following Lemma:

Lemma 2. *Let (b_C^*, β^*) be a RAE of the GSP with UC-restrictions. Then, there exists $\beta' \in B^*$ such that $\beta'(b_C^*) = \beta^*(b_C^*)$ and*

$$\begin{aligned}
& b_C^* \in \arg \max_{b_C} u_C(b_C, \beta'(b_C)) \\
\text{subject to : } & \text{(R')} \exists v'_C \in \mathbb{R}_+^{|\mathcal{C}|}, b_{-C} \in \mathbb{R}_+^{n-|\mathcal{C}|} \text{ s.t. } (b_C, \beta'(b_{-C})) \in \mathcal{EG}(v'_C, v_{-C}) \\
& \text{(S.2)} \forall i \in \mathcal{C}, u_i(b_C, b_{-C}^*) \geq \bar{u}_i^{\mathcal{C} \setminus \{i\}}.
\end{aligned}$$

Proof: First, note that if the coalition bids consistently with (R) and the independents in $\beta^*(b_C)$ are efficiently ordered, then the usual recursion of BR_i^* ensures that $(b_C, \beta^*(b_C))$ is a LREF equilibrium for values (v'_C, v_{-C}) . Unfortunately though, because b_C is fixed in the first stage, the recursion is broken if some independents switch positions, so that it is possible that the independents bid according to BR_i^* and yet the independent below a given coalition member is higher-valued than the independent below. (However, the recursion still applies for any contiguously placed blocks of independents, forcing efficient ordering within such blocks.) To rule out such a possibility in equilibrium, we must appeal to the recursive stability condition: Suppose we have proved efficiency for all \mathcal{C}' with $|\mathcal{C}'| < |\mathcal{C}|$. Note that, by the argument above, this also proves Theorem 2 for all such \mathcal{C}' . Now, consider the lowest-placed member of the full coalition \mathcal{C} . Now, as in the proof of Theorem 1, let i denote the lowest-bidding coalition member, and let r denote the position of his bid in the overall ranking. If the $n - r$ independents below i are in fact the $n - i$ lowest-valued independents, then, just as in (16), the lowest coalition member, i , just obtains her outside option, $u_i^{\mathcal{C} \setminus \{i\}}$. But, just as in the proof of Theorem 1, i fails to obtain her outside option if the number of independents

below is not equal to $n - i$: Observe that $\sum_{k=i+1}^{S+1} v'_k(x^{k-1} - x^k) > \sum_{k=i+1}^{S+1} v_k(x^{k-1} - x^k)$ unless $v_k = v'_k$ for all $k \geq i + 1$, so that the payoff of i is strictly less than $u_i^{C \setminus \{i\}}$ if the independents below are not the lowest-valued. Thus, the recursive stability condition (S.2) requires that the independents below the lowest coalition member are efficiently ordered with respect to the rest. Repeating this argument for all coalition members other than the highest shows that all independents below the second-highest-placed coalition member are efficiently ordered with respect to each other and with respect to those above. Finally, since the highest-placed coalition member is simply maximizing her own payoff, she cannot be placed below a lower-valued independent (since she could increase her payoff by raising her bid), or above a higher-valued independent (since she could increase her payoff by lowering her bid). This, together with the fact that independents are efficiently ordered among themselves, completes the the proof that all independents are efficiently ordered and therefore that $(b_C, \beta^*(b_C))$ forms a LREF equilibrium for some v'_C . This concludes the proof of the Lemma. ■

Armed with this Lemma, if b^* is a UC-RAE bid profile, then there exists beliefs β^* such that (b^*, β^*) solves the equilibrium conditions replacing constraint (R) with constraint (R'), hence such beliefs β^* necessarily satisfy the condition $(b_C, \beta^*(b_C)) \in \mathcal{EG}(v'_C, v_{-C})$ for all *valid* $b_C \in R_C^{UC}$ (i.e., for all $b_C \in R_C^{UC}$ satisfying constraint (S.2))

To fix terminology, consider the following equivalence relation \sim on \mathbb{R}_+^n : let $v \sim v'$ if and only if the following two conditions hold: (1) $\arg \max_{i \in I} v_i = \arg \max_{i \in I} v'_i$; (2) $v_i = v'_i$ for all $i \neq \arg \max_{i \in I} v_i$. That is, $v \sim v'$ if the profiles v and v' only differ in the highest valuation, but not in the identity of the highest valuation individual. For any $v \in \mathbb{R}_+^n$, let $[v]$ denote the equivalence class of v under this equivalence relation, and let \mathbb{V}^\sim denote the set of such equivalence classes. Next, consider the LREF equilibrium correspondence $\mathcal{EG} : \mathbb{R}_+^n \rightrightarrows \mathbb{R}_+^n$, which assigns to each profile $v \in \mathbb{R}_+^n$ the set $\mathcal{EG}(v)$ of LREF equilibrium profiles in the GSP auction (without coalition). Denote the set of equivalence classes under \sim on the range of \mathcal{EG} as $\mathcal{EG}(\mathbb{V}^\sim) \subseteq \mathbb{V}^\sim$, and let $\mathcal{EG}^\sim : \mathbb{V}^\sim \rightarrow \mathcal{EG}(\mathbb{V}^\sim)$ denote the function induced by \mathcal{EG} . Lemma 1 implies that \mathcal{EG}^\sim is a bijection. Further note that the payoffs of all bidders in the GSP with bids $\mathcal{EG}(v)$ are the same as in the VCG with truthful bids:

$$\text{for all } v \in \mathbb{R}_+^n \text{ and } i \in I, u_i^Y(v) = u_i^G(\mathcal{EG}(v)). \quad (17)$$

Since \mathcal{EG}^\sim is a well-defined function on the equivalence classes of \sim , v'_c in the restriction (R') uniquely pins down (b_c, b_{-c}^*) up to the highest overall bid, i.e., $(b_c, b_{-c}^*), (b'_c, b'_{-c}) \in \mathcal{EG}(v'_c, v_{-c})$ implies $(b_c, b_{-c}^*) \sim (b'_c, b'_{-c})$. Together with (17), this in turn implies that $u_i(b_c, b_{-c}^*) = u_i^\mathcal{V}(v'_c, v_{-c})$, so that also $u_c(b_c, b_{-c}^*) = u_c^\mathcal{V}(v'_c, v_{-c})$. As a result, we can now easily recast the coalition's problem as one of choosing v'_c (the coalition's 'feigned valuations'):

$$\begin{aligned} & \max_{v'_c} u_c^\mathcal{V}(v'_c, v_{-c}) \\ \text{subject to} & : \text{ (R')} (b_c, \beta^*(b_{-c}^*)) \in \mathcal{EG}(v'_c, v_{-c}). \\ & : \text{ (S.2)} \forall i \in \mathcal{C}, u_i^\mathcal{V}(v'_c, v_{-c}) \geq \bar{u}_i^{\mathcal{C} \setminus \{i\}}. \end{aligned}$$

Notice that (R') is irrelevant to the optimal choice of v'_c . Furthermore, $\bar{u}_i^{\mathcal{C}} = \bar{u}_i^{\mathcal{C}; \mathcal{V}}$ for all i when $|\mathcal{C}| = 1$, and the recursion defining $\bar{u}_i^{\mathcal{C}}$ is identical to that defining $\bar{u}_i^{\mathcal{C}; \mathcal{V}}$. It follows that the coalition's problem is now equivalent to its problem in the VCG game. By Theorem 1, the solution $v_c'^*$ is unique up to the report of the highest coalition member, $v_{\min(\mathcal{C})}'^*$.

Finally, by (R'), the RAE of the GSP $(b_c^*, \beta^*(b_{-c})) \in \mathcal{EG}(v_c'^*, v_{-c})$. Hence all bidders' positions and payoffs in this GSP equilibrium are therefore the same as in the unrestricted RAE of the VCG, $(v_c'^*, v_{-c})$. Because the ordering of bidders in the RAE of the VCG is efficient (Theorem 1), so is the ordering of bidders in the the RAE of the GSP. However, because $v_c'^*$ is unique only up to the highest coalition bid, $(b_c^*, \beta^*(b_{-c}))$ is not uniquely defined: there exists a continuum of equilibria differing in the payments of all bidders above the highest coalition bidder: for each $v_{\min(\mathcal{C})}'^* \in (v_{\min(\mathcal{C})+1}'^*, v_{\min(\mathcal{C})-1}'^*)$, there exists one equivalence class of RAE of the GSP, $[(b_c^*, \beta^*(b_{-c}))]$. Because \mathcal{EG} is unique only up to \sim (i.e., up to the highest overall bid), there also exist a continuum of equilibria yielding the same payoffs and positions, but differing in the highest overall bid, within each $[b^*]$. In this sense, the equilibrium is unique up to the highest coalition and overall bids.

A.2.4 Proof of Theorem 3

The claim about the possibility of strict ordering in revenues is proven by Example 5 in the text. Here we prove the general claims about existence, uniqueness and weak ordering. The proof is by construction, and it is based on the following intermediate result.

Lemma 3. Fix $\mathcal{C} \subset I$, and let \mathcal{K} be a finite indexing set. Let $\{b^{(k)}\}_{k \in \mathcal{K}}$ be a collection of bid profiles such that, for each $k \in \mathcal{K}$, $b_{-\mathcal{C}}^{(k)} \in BR_{-\mathcal{C}}^*(b_{\mathcal{C}}^{(k)})$ and $\rho(i; b^{(k)}) = i$ for each $i \in I$. Define $\mathcal{L}(\{b^{(k)}\}_{k \in \mathcal{K}}) \equiv \hat{b} \in \mathbb{R}_+^n$ as follows:

$$\hat{b}_i = \begin{cases} \hat{b}_i = \min_{k \in \mathcal{K}} b_i^{(k)} & \text{if } i \in \mathcal{C} \\ \hat{b}_i = v_{S+1} & \text{if } i = S+1 \notin \mathcal{C} ; \\ \frac{1}{x^{i-1}} \left[\sum_{j=i}^{\bar{c}(i)-1} v_j(x^{j-1} - x^j) + \hat{b}_{\bar{c}(i)} x^{\bar{c}(i)-1} \right] & \text{otherwise} \end{cases}$$

where $\bar{c}(i) := \min \{j \in \mathcal{C} \mid j > i\}$ if $i < \max \mathcal{C}$ and $\bar{c}(i) = S+1$ otherwise.

Then: (i) $\rho(i; \hat{b}) = i \quad \forall i \in I$; (ii) $u_i(\hat{b}) \geq u_i(b^{(k)})$ for all $i \in I$ and for all $k \in \mathcal{K}$, with strict inequality whenever $\hat{b}_{\bar{c}(i)} \neq b_{\bar{c}(i)}^{(k)}$; (iii) $u_{\mathcal{C}}(\hat{b}) \geq u_{\mathcal{C}}(b^{(k)})$ for all $k \in \mathcal{K}$, with strict inequality whenever $\exists i \in \mathcal{C} \setminus \min \mathcal{C}$ such that $b_i^{(k)} \neq \hat{b}_i$; (iv) $\hat{b}_{-\mathcal{C}} \in BR_{-\mathcal{C}}^*(\hat{b}_{\mathcal{C}})$.

Proof of Lemma 3

We begin by noting that because for each $k \in \mathcal{K}$, $b_{-\mathcal{C}}^{(k)} \in BR_{-\mathcal{C}}^*(b_{\mathcal{C}}^{(k)})$ and $\rho(i; b^{(k)}) = i$ for each $i \in I$, we have that $\forall k \in \mathcal{K}, i \notin \mathcal{C}$ s.t. $i \neq S+1$,

$$b_i^{(k)} = \frac{1}{x^{i-1}} \left[\sum_{j=i}^{\bar{c}(i)-1} v_j(x^{j-1} - x^j) + b_{\bar{c}(i)}^{(k)} x^{\bar{c}(i)-1} \right],$$

and $b_i^{(k)} = v_{S+1}$ if $i = S+1 \notin \mathcal{C}$ ($\bar{c}(i)$ is defined in the statement in the Lemma) The following two key observations are now immediate:

1. For every $k \in \mathcal{K}$ and for every $i \in I$, $\hat{b}_i \leq b_i^{(k)}$: For $i \in \mathcal{C}$, $\hat{b}_i \leq b_i^{(k)}$ by the definition of coalition bids in the statement of the lemma. For $i = S+1 \notin \mathcal{C}$, $\hat{b}_i = v_{S+1} = b_i^{(k)}$ (the second equality is because the Lemma requires $b_{-\mathcal{C}}^{(k)} \in BR_{-\mathcal{C}}^*(b_{\mathcal{C}}^{(k)})$) Since, by definition $\bar{c}(i) \in \mathcal{C} \cup \{S+1\}$, it follows that $\hat{b}_{\bar{c}(i)} \leq b_{\bar{c}(i)}^{(k)}$. Finally, for $i \notin \mathcal{C}$ s.t. $i \neq S+1$,

$$\hat{b}_i = \frac{1}{x^{i-1}} \left[\sum_{j=i}^{\bar{c}(i)-1} v_j(x^{j-1} - x^j) + \hat{b}_{\bar{c}(i)} x^{\bar{c}(i)-1} \right] \leq \frac{1}{x^{i-1}} \left[\sum_{j=i}^{\bar{c}(i)-1} v_j(x^{j-1} - x^j) + b_{\bar{c}(i)}^{(k)} x^{\bar{c}(i)-1} \right] = b_i^{(k)},$$

because $\hat{b}_{\bar{c}(i)} \leq b_{\bar{c}(i)}^{(k)}$. Note that the inequality is strict whenever $\hat{b}_{\bar{c}(i)} \neq b_{\bar{c}(i)}^{(k)}$.

2. For each $i \in I$, there exists $k \in \mathcal{K}$ such that $b_i = b_i^{(k)}$. For $i \in \mathcal{C}$ this is immediate from the definition. For $i = S + 1 \notin \mathcal{C}$, $\hat{b}_i = v_{S+1} = b_i^{(k)}$ for all k (cf. previous point). For $i \notin \mathcal{C}$ s.t. $i \neq S + 1$, the result follows because $\bar{c}(i) \in \mathcal{C} \cup \{S + 1\}$, so that, by the result we just established, there exists $k \in \mathcal{K}$ such that $\hat{b}_{\bar{c}(i)} = b_{\bar{c}(i)}^{(k)}$, so that

$$\hat{b}_i = \frac{1}{x^{i-1}} \left[\sum_{j=i}^{\bar{c}(i)-1} v_j(x^{j-1} - x^j) + \hat{b}_{\bar{c}(i)} x^{\bar{c}(i)-1} \right] = \frac{1}{x^{i-1}} \left[\sum_{j=i}^{\bar{c}(i)-1} v_j(x^{j-1} - x^j) + b_{\bar{c}(i)}^{(k)} x^{\bar{c}(i)-1} \right] = b_i^{(k)},$$

We can now easily establish the lemma's results:

(i) $\rho(i; \hat{b}) = i$ for all $i \in I$: Let $i, j \in I$ with $i < j$ (so that $v_i > v_j$). We must prove that $\hat{b}_i > \hat{b}_j$. Now, by point 2 above, there exists $k \in \mathcal{K}$ such that $\hat{b}_i = b_i^{(k)}$. Because $b^{(k)}$ is efficiently ordered, $b_i^{(k)} > b_j^{(k)}$. Finally, by point 1, $b_j^{(k)} \geq \hat{b}_j$. Hence, $\hat{b}_i = b_i^{(k)} > b_j^{(k)} \geq \hat{b}_j$, as desired.

(ii) $u_i(\hat{b}) \geq u_i(b^{(k)})$ for all $i \in I$ and all $k \in \mathcal{K}$, with strict inequality if $\hat{b}_{\bar{c}(i)} \neq b_{\bar{c}(i)}^{(k)}$: Because i obtains its efficient position under both \hat{b} (established in (i)) and $b^{(k)}$ (given), we have

$$u_i(\hat{b}) = (v_i - \hat{b}_{i+1})x^i \geq (v_i - b_{i+1}^{(k)})x^i = u_i(b^{(k)}),$$

where the inequality holds because $\hat{b}_{i+1} \leq b_{i+1}^{(k)}$ by point 1 above, with strict inequality if $\hat{b}_{\bar{c}(i)} \neq b_{\bar{c}(i)}^{(k)}$, as noted at the end of point 1.

(iii) $u_c(\hat{b}) \geq u_c(b^{(k)})$ for all $k \in \mathcal{K}$, with strict inequality whenever $\exists i \in \mathcal{C} \setminus \min \mathcal{C}$ such that $b_i^{(k)} \neq \hat{b}_i$: The weak inequality follows immediately from part (ii). Now, suppose $b_i^{(k)} \neq \hat{b}_i$ for some $i \in \mathcal{C} \setminus \min \mathcal{C}$, and let $j = \max \{k \in \mathcal{C} | k < i\}$ be the coalition member directly above i in the ranking of valuations. Then $\bar{c}(j) = i$, so that by the strict inequality part of result (ii), $u_j(b^{(k)}) < u_j(\hat{b})$. Since $u_{j'}(b^{(k)}) \leq u_{j'}(\hat{b})$ for all other terms in the sums defining $u_c(\cdot)$, this completes the proof for strict inequality.

(iv) $\hat{b}_{-c} \in BR_{-c}^*(\hat{b}_{-c})$: The LREF condition holds by construction. We must simply prove the Nash condition, i.e., that each $i \notin \mathcal{C}$ (weakly) prefers position i to position j for all $j \in I$. Define $j' = j + 1$ if $j > i$ and $j' = j$ if $j < i$. Note that if bidder i deviates to position $j \neq i$ under bid profile \hat{b} , it gets payoff $(v_i - \hat{b}_{j'})x^j$. By the observation in point 2 above, there exists some k such that $\hat{b}_{j'} = b_{j'}^{(k)}$, so that $(v_i - \hat{b}_{j'})x^j = (v_i - b_{j'}^{(k)})x^j$. Because

$b_{-c}^{(k)} \in BR_{-c}^*(b_{-c}^{(k)})$ and $\rho(i; b^{(k)}) = i$, i cannot profitably deviate from position i to position $j \neq i$ under bid profile $b^{(k)}$, i.e. $(v_i - b_{j'}^{(k)})x^j \leq (v_i - b_{i+1}^{(k)})x^i$. Finally, by point 1 above, $b_{i+1}^{(k)} \geq \hat{b}_{i+1}$, so that $(v_i - b_{i+1}^{(k)})x^i \leq (v_i - \hat{b}_{i+1})x^i$. Putting these results together,

$$(v_i - \hat{b}_{i+1})x^i \geq (v_i - b_{i+1}^{(k)})x^i \geq (v_i - b_{j'}^{(k)})x^j = (v_i - b_{j'})x^j.$$

That is, bidder i cannot profitably deviate to position $j \neq i$ under bid profile \hat{b} , as desired. This concludes the proof of the Lemma. ■

Armed with this Lemma, we can now prove Theorem 3. We begin with existence and weak ordering of revenues, using induction on the coalition's size, C . For the **induction basis**, we use $C = 1$. Both existence and weak order now hold trivially, as both the efficiency-constrained RAE of the GSP and the RAE of the VCG mechanism are equal to the LREF equilibrium by definition.

For the **inductive step**, suppose that for all coalitions of size $C - 1$ Eff-RAE exist, then we show that Eff-RAE also exists for any coalition of size C , and that in each of these RAE the coalition's surplus is no lower than in any RAE of the VCG mechanism, while the auctioneer's revenue is no higher than in a corresponding RAE of the VCG mechanism.

Fix \mathcal{C} with $|\mathcal{C}| = C$. Let $b^{UC} \in \mathbb{R}_+^n$ be the bids in the UC-RAE of the GSP auction with the same coalition \mathcal{C} , in which the top coalition member is placing the highest possible bid (this exists and is unique by Theorem 2). Observe that because of the bijection between UC-RAE of the GSP auction and unconstrained RAE of the VCG mechanism (established in Theorem 2), we can use the coalition's surplus in the GSP auction with bids b^{UC} as our reference point. Next, note that, for any $b_{\mathcal{C}}$, the beliefs $\beta^*(b_{\mathcal{C}})$ in any Eff-RAE of the GSP auction are uniquely determined by the Varian/EOS recursion. Hence, a complete Eff-RAE, $(b^*, \beta^*) \in \mathbb{R}_+^n \times B^*$, if it exists, is in fact fully determined by $b_{\mathcal{C}}^* \in \mathbb{R}_+^C$. We now proceed to prove that such a $b_{\mathcal{C}}^*$ exists by constructing a candidate profile.

For each $i \in \mathcal{C}$, let $b^{(i)}$ be the bids in an Eff-RAE with coalition $\mathcal{C} \setminus \{i\}$ (these exist under the inductive hypothesis). Let $b^{(0)} = b^{UC}$. Let

$$\hat{b} = \mathcal{L} \left(\{b^{(i)}\}_{i \in \mathcal{C} \cup \{0\}} \right),$$

where \mathcal{L} is as defined in Lemma 3. Now, by results (i) and (iv) of Lemma 3, we have $\rho(i; \hat{b}) = i$ for all $i \in I$ and $\hat{b}_{-C} \in BR_{-C}^*(\hat{b}_C)$. It follows that $\hat{b}_C \in R_C^{EFF}$. By result (ii) of Lemma 3, $u_i(\hat{b}) \geq u_i(b^{(k)})$ for each i . Moreover, by construction, $u_i(b^{(k)}) = \bar{u}_i^{C \setminus \{i\}}$ for each $i \in C$, hence profile \hat{b} satisfies the recursive stability condition. It follows that \hat{b}_C is a valid bid vector for coalition C trying to achieve an Eff-RAE and that $\hat{b}_{-C} = \beta^*(\hat{b}_C)$, where β^* are the unique beliefs consistent with Eff-RAE. Maintaining the assumption of finite bid increments, as in Theorems 1 and 2, the coalition is therefore maximizing over a non-empty, finite set of valid bid vectors, so that a maximum, b_C^* , exists. Thus, an efficiency constrained RAE for coalition C exists (and is equal to $((b_C^*, \beta^*(b_C^*)), \beta^*)$).

Now the weak ordering of coalition surplus is immediate: Result (iii) of Lemma 3 implies $u_C(\hat{b}) \geq u_C(b^{UC})$, and clearly the optimal bid profile $(b_C^*, \beta^*(b_C^*))$ must satisfy $u_C(b_C^*, \beta^*(b_C^*)) \geq u_C(\hat{b})$. It follows that $u_C(b_C^*, \beta^*(b_C^*)) \geq u_C(b^{UC})$.

Next, we establish the ordering for the auctioneer's revenues. We first show that, in the Eff-RAE (b^*, β^*) , the bid of coalition members other than the highest-valuation is weakly lower than in \hat{b} . To this end, suppose that there exists some $i \in C \setminus \min C$ such that $b_i^* > \hat{b}_i$. Let $b' = \mathcal{L}(\{b^*, \hat{b}\})$. By part (i) of Lemma 3, b'_C is still a valid bid vector for the coalition, whereas part (iii) implies $u_C(b'_C, \beta^*(b'_C)) > u_C(b_C^*, \beta^*(b_C^*))$ which contradicts the optimality of b_C^* . We thus conclude that $b_i^* \leq \hat{b}_i$ for all $i \in C \setminus \min C$.

Because the independents' bids are fixed by the recursion under both \hat{b} and b^* , we know that in fact $b_i^* \leq \hat{b}_i$ for all $i > \min C$. Because by construction $\hat{b}_i \leq b_i^{UC}$ for all $i \in I$, we thus have $b_i^* \leq b_i^{UC}$ for all $i > \min C$. If $\min C = 1$, this completes the proof that the auctioneer's revenues are weakly lower under b^* than under b^{UC} . If $\min C > 1$, we need to show that even the top coalition bidder in b^* cannot bid more than this bidder's maximum possible UC-RAE bid. Because $b_{\min C}^{UC}$ is the maximum bid that the top coalition bidder can place in a UC-RAE, it is equal to (cf. Theorem 2)

$$b_{\min C}^{UC} = v_{\min C-1} - \frac{x^{\min C}}{x^{\min C-1}} (v_{\min C-1} - b_{\min C+1}^{UC}).$$

If $b_{\min \mathcal{C}}^* > b_{\min \mathcal{C}}^{UC}$, then the independent above the top coalition member obtains a payoff

$$U_0 = (v_{\min \mathcal{C}-1} - b_{\min \mathcal{C}}^*)x^{\min \mathcal{C}-1} < (v_{\min \mathcal{C}-1} - b_{\min \mathcal{C}}^{UC})x^{\min \mathcal{C}-1} = (v_{\min \mathcal{C}-1} - b_{\min \mathcal{C}+1}^{UC})x^{\min \mathcal{C}},$$

where the last inequality follows by substituting in the expression for $b_{\min \mathcal{C}}^{UC}$ from above.

Dropping one position down this independent would obtain

$$U' = (v_{\min \mathcal{C}-1} - b_{\min \mathcal{C}+1}^*)x^{\min \mathcal{C}} \geq (v_{\min \mathcal{C}-1} - b_{\min \mathcal{C}+1}^{UC})x^{\min \mathcal{C}} > U_0,$$

where the first inequality follows because $b_i^* \leq b_i^{UC}$ for all $i > \min \mathcal{C}$, as established above. Thus this independent has a profitable deviation; a contradiction. We conclude that $b_{\min \mathcal{C}}^* \leq b_{\min \mathcal{C}}^{UC}$. But then, by the independents' recursion, we also have $b_i^* \leq b_i^{UC}$ for all $i \leq \min \mathcal{C}$. Because we already knew that the $b_i^* \leq b_i^{UC}$ for all $i > \min \mathcal{C}$, we have established that all bids in b^* are weakly lower than in b^{UC} , which completes the claim about the auctioneer's revenues.

Next, we show that the Eff-RAE is unique up to the highest coalition bid. To this end, fix some coalition $\mathcal{C} \subseteq I$ and let b^{R1} and b^{R2} be two (possibly equal) Eff-RAE for \mathcal{C} . Let $\hat{b} := \mathcal{L}(\{b^{R1}, b^{R2}\})$.

By results (i), (iii) and (iv) of Lemma 3, \hat{b} is still efficiently ordered and $\hat{b}_{-\mathcal{C}} \in BR_{-\mathcal{C}}^*(\hat{b}_{\mathcal{C}})$, so that $\hat{b}_{\mathcal{C}}$ is in the set of permitted bids for the coalition in the efficiency-constrained problem without the recursive stability restriction, with $\hat{b}_{-\mathcal{C}} \in \beta^*(\hat{b}_{\mathcal{C}})$.

Furthermore, by result (ii) of Lemma 3, each coalition member is at least as well off under \hat{b} as under b^{R1} and b^{R2} . Therefore, the fact that b^{R1} and b^{R2} satisfy the recursive stability condition implies that so does \hat{b} . The optimality of $b_{\mathcal{C}}^{R1}$ and $b_{\mathcal{C}}^{R2}$ in this set therefore implies that $u_{\mathcal{C}}(\hat{b}) \leq u_{\mathcal{C}}(b^{Rk}) \forall k \in \{1, 2\}$. But result (iii) of Lemma 3 then implies that $\hat{b}_i = b_i^{R1} = b_i^{R2}$ for all $i \in \mathcal{C} \setminus \min \mathcal{C}$.

Combining these results yields $b_i^{R1} = b_i^{R2} = \hat{b}_i$ for all $i \in \mathcal{C} \setminus \min \mathcal{C}$. Because coalition bids also uniquely determine independents' bids, the Eff-RAE is thus unique up to the highest coalition bid. This completes the proof.

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