Sincere Voting and Information Aggregation with Voting Costs*

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August 2007

Abstract

We study the properties of equilibrium voting in two-alternative elections under the majority rule. Voters are privately informed about who is the "better" candidate and have private costs of going to the polls. We show that sincere voting is always an equilibrium. Furthermore, even though the equilibrium percentage turnout goes to zero in the limit, information fully aggregates under the sincere voting equilibrium.

1 The Model

There are two candidates, named A and B, who are competing in an election. The winner needs a simple majority of the votes cast. In the event of a tied vote, the winning candidate is chosen by a fair coin toss. There are two equally likely states of nature, α and β. Candidate A is the better candidate in state α while candidate B is the better candidate in state β. Specifically, in state α the payoff of any voter is 1 if A is elected and 0 if B is elected. In state β, the roles of A and B are reversed.

The size of the electorate is a random variable which is distributed according to a Poisson distribution with mean n. Thus the probability that there are exactly m eligible voters (or *citizens*) is

$$g_n(m) = \frac{e^{-n}n^m}{m!}$$

Prior to voting, every citizen receives a private signal S_i regarding the true state of nature. The signal can take on one of two values, a or b. The probability of receiving a particular signal depends on the true state of nature. Specifically,

$$\Pr[a \mid \alpha] = r_\alpha \text{ and } \Pr[b \mid \beta] = r_\beta$$

We suppose that both r_α and r_β are greater $\frac{1}{2}$, so that the signals are informative and also that both are less than 1, so that they are noisy. Thus, signal a is associated with state α while the signal b is associated with β.

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*Preliminary draft. Comments welcome.

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Conditional on the state of nature, the signals of the voters are realized independently. The posterior probabilities of the states after receiving signals are

$$q(\alpha \mid a) = \frac{r_\alpha}{r_\alpha + (1 - r_\beta)} \text{ and } q(\beta \mid b) = \frac{r_\beta}{r_\beta + (1 - r_\alpha)}$$

We assume, without loss of generality, that $r_\alpha \geq r_\beta$. It may be verified that

$$q(\alpha \mid a) \leq q(\beta \mid b)$$

Thus the posterior probability of state is α given the signal a is smaller than the posterior probability of state β given the signal b even though the "correct" signal is more likely in state α.

Of particular interest are circumstances where voting is *sincere*—where, in equilibrium, all those who receive a signal of a vote for A and those who receive a signal of b vote for B.

1.0.1 Compulsory Voting

First, consider the case where all those in the electorate must vote for either A or B; that is, no abstentions are allowed. When the signal precisions are equal (that is, $r_\alpha = r_\beta$), it is easy to see that sincere voting is indeed an equilibrium. Consider a voter with signal a and who votes sincerely. Her payoffs are only affected when her vote is *pivotal* (that is, changes the outcome of the election). When the election is in a tie, there are an equal number of a and b signals recevied by the other voters and, since signal precisions are equal, these signals cancel each other out and there is no net informational effect on the voter's posterior beliefs. Since her own signal favors state α, it pays to vote sincerely. Next, consider the case where A trails by a single vote. In that case, there is one more b signal than a signal from the other voters, and the informational effect of this signal is exactly cancelled by the voter's own a signal. In other words, the posterior beliefs of the voter in this circumstance are equal to her priors and hence she is indifferent between voting for A and voting for B. Thus, it pays to vote sincerely if the signal precisions are equal.

When the signal precisions are not equal, say $r_\alpha > r_\beta$, sincere voting does not constitute an equilibrium. Once again, to determine the optimality of voting sincerely, a voter needs to consider only circumstances in which he or she is *pivotal.* Specifically, let Piv_A denote the set of circumstances in which a single vote for A will change the outcome of the election in favor of A. The probability of the event Piv_A in state α is

$$\Pr[Piv_A \mid \alpha] = \frac{1}{2}\sum_{k=0}^{\infty} e^{-n}\frac{(nr_\alpha)^k (n(1-r_\alpha))^k}{k!k!} + \frac{1}{2}\sum_{k=0}^{\infty} e^{-n}\frac{(nr_\alpha)^k (n(1-r_\alpha))^{k+1}}{k!k+1!}$$

The first expression pertains to situation in which there is a tie (so that A is winning with probability $\frac{1}{2}$) and a single vote for A will result in A winning for sure. The second expression pertains to situations in which A is losing by exactly one vote and a single vote for A will throw the election into a tie. Myerson (1998) has shown that, when n is large, the "pivot probabilities" for candidate A in the two states α and β can be approximated by the formulae

$$\Pr\left[Piv_A \mid \alpha\right] \approx \frac{e^{n\left(2\sqrt{r_\alpha(1-r_\alpha)}-1\right)}}{4\sqrt{\pi n\sqrt{r_\alpha\left(1-r_\alpha\right)}}}\left(1+\sqrt{\frac{1-r_\alpha}{r_\alpha}}\right)$$

$$\Pr\left[Piv_A \mid \beta\right] \approx \frac{e^{n\left(2\sqrt{r_\beta(1-r_\beta)}-1\right)}}{4\sqrt{\pi n\sqrt{r_\beta\left(1-r_\beta\right)}}}\left(1+\sqrt{\frac{r_\beta}{1-r_\beta}}\right)$$

Similarly, let Piv_B denote the set of events in which a single vote for B will change the outcome of the election in favor of B. Using the same approximation formulae, we obtain

$$\Pr\left[Piv_B \mid \alpha\right] \approx \frac{e^{n\left(2\sqrt{r_\alpha(1-r_\alpha)}-1\right)}}{4\sqrt{\pi n\sqrt{r_\alpha\left(1-r_\alpha\right)}}}\left(\frac{\sqrt{r_\alpha}+\sqrt{1-r_\alpha}}{\sqrt{1-r_\alpha}}\right)$$

$$\Pr\left[Piv_B \mid \beta\right] \approx \frac{e^{n\left(2\sqrt{r_\beta(1-r_\beta)}-1\right)}}{4\sqrt{\pi n\sqrt{r_\beta\left(1-r_\beta\right)}}}\left(\frac{\sqrt{1-r_\beta}+\sqrt{r_\beta}}{\sqrt{r_\beta}}\right)$$

For sincere voting to be optimal, it must be the case that, after receiving signal a, a voter's expected payoff from voting for A exceeds her payoff from voting for B. Formally, this amounts to the condition

$$\begin{aligned} & q\left(\alpha \mid a\right)\Pr\left[Piv_A \mid \alpha\right]-q\left(\beta \mid a\right)\Pr\left[Piv_A \mid \beta\right] \\ \geq\; & q\left(\beta \mid a\right)\Pr\left[Piv_B \mid \beta\right]-q\left(\alpha \mid a\right)\Pr\left[Piv_B \mid \alpha\right] \end{aligned}$$

Rearranging terms, the "incentive compatibility" constraint becomes

$$q\left(\alpha \mid a\right)\left(\Pr\left[Piv_A \mid \alpha\right]+\Pr\left[Piv_B \mid \alpha\right]\right) \geq q\left(\beta \mid a\right)\left(\Pr\left[Piv_A \mid \beta\right]+\Pr\left[Piv_B \mid \beta\right]\right)$$

Next, we compare the probability of being pivotal in state α compared to state β. When n is large, the ratio of pivot probabilities is

$$\frac{\Pr\left[Piv_A \mid \alpha\right]+\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]+\Pr\left[Piv_B \mid \beta\right]} \approx e^{n\left(2\sqrt{r_\alpha(1-r_\alpha)}-2\sqrt{r_\beta(1-r_\beta)}\right)}k\left(r_\alpha, r_\beta\right)$$

where $k\left(r_\alpha, r_\beta\right)$ is positive and is independent of n. Since $r_\alpha > r_\beta > \frac{1}{2}$, $r_\beta\left(1-r_\beta\right) > r_\alpha\left(1-r_\alpha\right)$ and so

$$\lim_{n\to\infty}\frac{\Pr\left[Piv_A \mid \alpha\right]+\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]+\Pr\left[Piv_B \mid \beta\right]} = 0$$

This implies that, when n is large and a voter is pivotal, state β is infinitely more likely than is state α. Thus, the required inequality for the incentive compatibility condition does not hold and hence a voter with an a signal will not wish to vote sincerely in a large election. To summarize, we have shown,

Proposition 1 *Under compulsory voting, sincere voting is not an equilibrium in large elections with different signal precisions ($r_\alpha \neq r_\beta$).*

1.1 Costly Voting

Up until now, we have assumed every voter cast a vote for either A or B; that is, abstention was not allowed. In reality, of course, abstention is always possible. A second aspect is that voters incur some costs of going to the polls to cast their votes. For the remainder of this paper, we add simultaneously introduce the possiblity of abstention and voting costs. We suppose that voting costs vary across voters. Specifically, the cost of voting for each voter is private information and determined by a realization from a continuous probability distribution F with support $[0, \overline{c}]$. We suppose that $\overline{c} \geq \frac{1}{2}$ and that F admits a density f that is strictly positive on $(0, \overline{c})$.

Finally, we assume that voting costs are independently distributed across voters and independent of the signal as to who is the better candidate. Thus prior to the voting decision, each voter has two pieces of private information—his cost of voting and a signal regarding the state.

2 Equilibrium with Costly Voting

We will show that under majority rule, there exists an equilibrium of the voting game with the following features.

1. There exists a pair of cut-off cost levels, c_a and c_b, such that a citizen with a cost realization c and who receives a signal $i = a, b$ votes if and only if $c \leq c_i$.

2. All those who vote do so sincerely—that is, all those with a signal of a vote for A and those with a signal of b vote for B. Let us denote by p_a the probability that a voter with signal a will actually vote, so that $p_a = F(c_a)$. Similarly, $p_b = F(c_b)$.

Consider an event where (other than voter 1) the realized electorate is of size m and there are k votes in favor of A and l votes in favor of B. The number of abstentions is thus $m - k - l$. The probability of this event in state α is

$$\begin{aligned}\Pr[\langle k, l; m\rangle \mid \alpha] &= \frac{e^{-n} n^m}{m!} \binom{m}{k+l} \binom{k+l}{k} \\ &\quad \times (1 - r_\alpha p_a - (1 - r_\alpha) p_b)^{m-k-l} (r_\alpha p_a)^k ((1 - r_\alpha) p_b)^l\end{aligned}$$

Notice that in the formula above, the probability of voting for A depends jointly on receiving signal a (which occurs with probability r_α) and a sufficiently low cost realization (which occurs with probability p_a). The probability of voting for b depends jointly on receiving signal b (which occurs with probability $1 - r_\alpha$) and a sufficiently low cost realization (which occurs with probability p_b). The complement of these two is the probability of abstention.

It is useful to rearrange $\Pr[\langle k, l; m\rangle \mid \alpha]$ as follows:

$$\begin{aligned}\Pr[\langle k, l; m\rangle \mid \alpha] &= \frac{e^{-n(1 - r_\alpha p_a - (1 - r_\alpha) p_b)}}{(m - k - l)!} (n (1 - r_\alpha p_a - (1 - r_\alpha) p_b))^{m-k-l} \\ &\quad \times \frac{e^{-n r_\alpha p_a}}{k!} (n r_\alpha p_a)^k \frac{e^{-n(1 - r_\alpha) p_b}}{l!} (n (1 - r_\alpha) p_b)^l\end{aligned}$$

Of course, the size of the electorate is unknown to the voter at the time of the vote. The probability of the event that the vote totals are k and l, written $\langle k, l \rangle$, irrespective of the size of the electorate, is

$$\begin{aligned} \Pr\left[\langle k, l \rangle \mid \alpha\right] &= \sum_{m=k+l}^{\infty} \Pr\left[\langle k, l; m \rangle \mid \alpha\right] \\ &= \frac{e^{-nr_\alpha p_a} \left(nr_\alpha p_a\right)^k}{k!} \frac{e^{-n(1-r_\alpha)p_b} \left(n\left(1-r_\alpha\right)p_b\right)^l}{l!} \end{aligned} \tag{1}$$

whereas the probability of $\langle k, l \rangle$ in state β is

$$\Pr\left[\langle k, l \rangle \mid \beta\right] = \frac{e^{-n\left(1-r_\beta\right)p_a} \left(n\left(1-r_\beta\right)p_a\right)^k}{k!} \frac{e^{-nr_\beta p_b} \left(nr_\beta p_b\right)^l}{l!} \tag{2}$$

If we define Piv_A to be the set of events in which, by voting for A as opposed to staying home, a citizen can change the outcome of the election, then we have

$$\begin{aligned} \Pr\left[Piv_A \mid \alpha\right] &= \frac{1}{2}\sum_{k=0}^{\infty} \Pr\left[\langle k, k \rangle \mid \alpha\right] + \frac{1}{2}\sum_{k=0}^{\infty} \Pr\left[\langle k, k+1 \rangle \mid \alpha\right] \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \left(e^{-n(r_\alpha p_a + (1-r_\alpha)p_b)} \frac{\left(n^2 r_\alpha \left(1-r_\alpha\right) p_a p_b\right)^k}{k!k!}\right)\left(1 + \frac{n\left(1-r_\alpha\right)p_b}{k+1}\right) \end{aligned}$$

and $\Pr\left[Piv_A \mid \beta\right]$ is determined similarly. Similarly, define Piv_B to be the set of events where voting for B rather than staying at home changes the election outcome.

Participation Decisions Using this notation, the expected payoff to a voter with signal a and cost c who votes sincerely is

$$E\pi_a\left(c\right) = q\left(\alpha \mid a\right) \Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid a\right) \Pr\left[Piv_A \mid \beta\right] - c$$

and similarly, the expected payoff to a voter with signal b and cost c who votes sincerely is

$$E\pi_b\left(c\right) = q\left(\beta \mid b\right) \Pr\left[Piv_B \mid \beta\right] - q\left(\alpha \mid b\right) \Pr\left[Piv_B \mid \alpha\right] - c$$

We may use these expressions to characterize the participation behavior in the election. Specifically, in any equilibrium, it must be the case that the cost thresholds c_a, c_b (or equivalently, the probability thresholds p_a, p_b) solve

$$q\left(\alpha \mid a\right) \Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid a\right) \Pr\left[Piv_A \mid \beta\right] = F^{-1}\left(p_a\right) \tag{3}$$
$$q\left(\beta \mid b\right) \Pr\left[Piv_B \mid \beta\right] - q\left(\alpha \mid b\right) \Pr\left[Piv_B \mid \alpha\right] = F^{-1}\left(p_b\right) \tag{4}$$

Lemma 1 *There exists a solution* (p_a^*, p_b^*) *to equations* (3) *and* (4) *such that* $0 < p_a^* < 1$ *and* $0 < p_b^* < 1$.

Proof. Notice that, at any point $(0, p_b)$

$$\begin{aligned}\Pr[Piv_A \mid \alpha] &= \frac{1}{2}e^{-n(1-r_\alpha)p_b}\left(1+n\left(1-r_\alpha\right)p_b\right) \\ \Pr[Piv_A \mid \beta] &= \frac{1}{2}e^{-nr_\beta p_b}\left(1+nr_\beta p_b\right)\end{aligned}$$

We claim that $\Pr[Piv_A \mid \alpha] > \Pr[Piv_A \mid \beta]$. This follows from that fact that the function $g(x) = e^{-x}(1+x)$ is strictly decreasing for $x > 0$ and that $r_\beta > 1 - r_\alpha$. Hence, at $(0, p_b)$

$$q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] > 0 = F^{-1}(0)$$

since $q(\alpha \mid a) > \frac{1}{2}$.

Next, notice that at any point $(1, p_b)$

$$q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] < \tfrac{1}{2} \leq F^{-1}(1)$$

Thus, for any p_b, there exists a value of $p_a \in (0,1)$ solving equation (3).

An identical argument shows that, for any p_a, there exists a value of $p_b \in (0,1)$ solving equation (4).

Therefore, an interior solution to equations (3) and (4) exists. ■

Voting Decisions Next, we show that for large n, it is optimal for a single voter to vote sincerely given the participation thresholds determined by equations (3) and (4) and given that all other voters are voting sincerely.

Consider a voter with signal a who decides to cast a vote. If all others participate according to the threshold costs determined above and if voting, vote sincerely, it is optimal for a voter with signal a to vote for A if and only if

$$q(\alpha \mid a)\left(\Pr[Piv_A \mid \alpha] + \Pr[Piv_B \mid \alpha]\right) \geq q(\beta \mid a)\left(\Pr[Piv_A \mid \beta] + \Pr[Piv_B \mid \beta]\right) \tag{5}$$

Similarly, given that all others vote sincerely, it is optimal for a voter with signal b to vote for B if and only if

$$q(\beta \mid b)\left(\Pr[Piv_A \mid \beta] + \Pr[Piv_B \mid \beta]\right) \geq q(\alpha \mid b)\left(\Pr[Piv_A \mid \alpha] + \Pr[Piv_B \mid \alpha]\right) \tag{6}$$

Myerson (1998) has shown that for large n, the pivotal probabilities can be approximated by

$$\Pr[Piv_A \mid \alpha] \approx \frac{e^{n\left(2\sqrt{r_\alpha p_a(1-r_\alpha)p_b} - r_\alpha p_a - (1-r_\alpha)p_b\right)}}{4\sqrt{\pi n\sqrt{r_\alpha p_a\left(1-r_\alpha\right)p_b}}}\frac{\sqrt{r_\alpha p_a} + \sqrt{\left(1-r_\alpha\right)p_b}}{\sqrt{r_\alpha p_a}} \tag{7}$$

$$\Pr[Piv_A \mid \beta] \approx \frac{e^{n\left(2\sqrt{r_\beta p_a(1-r_\beta)p_b} - (1-r_\beta)p_a - r_\beta p_b\right)}}{4\sqrt{\pi n\sqrt{r_\beta p_a\left(1-r_\beta\right)p_b}}}\frac{\sqrt{\left(1-r_\beta\right)p_a} + \sqrt{r_\beta p_b}}{\sqrt{\left(1-r_\beta\right)p_a}} \tag{8}$$

$$\Pr\left[Piv_B \mid \alpha\right] \approx \frac{e^{n\left(2\sqrt{r_\alpha p_a (1-r_\alpha) p_b} - r_\alpha p_a - (1-r_\alpha) p_b\right)}}{4\sqrt{\pi n \sqrt{r_\alpha p_a \left(1-r_\alpha\right) p_b}}} \frac{\sqrt{r_\alpha p_a} + \sqrt{\left(1-r_\alpha\right) p_b}}{\sqrt{\left(1-r_\alpha\right) p_b}} \quad (9)$$

$$\Pr\left[Piv_B \mid \beta\right] \approx \frac{e^{n\left(2\sqrt{r_\beta p_a (1-r_\beta) p_b} - (1-r_\beta) p_a - r_\beta p_b\right)}}{4\sqrt{\pi n \sqrt{r_\beta p_a \left(1-r_\beta\right) p_b}}} \frac{\sqrt{\left(1-r_\beta\right) p_a} + \sqrt{r_\beta p_b}}{\sqrt{r_\beta p_b}} \quad (10)$$

Recall from equation (3) that any individual voting for A obtains a gross benefit of

$$q\left(\alpha \mid a\right) \Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid a\right) \Pr\left[Piv_A \mid \beta\right] > 0$$

Hence, at any p_a and p_b that satisfies (3),

$$\frac{\Pr\left[Piv_A \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]} > \frac{q\left(\beta \mid a\right)}{q\left(\alpha \mid a\right)}$$

We will show that, for large n, it is also the case at any p_a and p_b that satisfies (3),

$$\frac{\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_B \mid \beta\right]} > \frac{q\left(\beta \mid a\right)}{q\left(\alpha \mid a\right)}$$

and together, these inqualities imply that the incentive compatibility condition for signal a, (5), is satisfied.

Lemma 2 *When n is large, for all p_a and p_b,*

$$\frac{\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_B \mid \beta\right]} > \frac{\Pr\left[Piv_A \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]}$$

Proof. Using Myerson's approximation formulae (7) to (10), this is the same as

$$\frac{\frac{\sqrt{r_\alpha p_a} + \sqrt{(1-r_\alpha) p_b}}{\sqrt{(1-r_\alpha) p_b}}}{\frac{\sqrt{(1-r_\beta) p_a} + \sqrt{r_\beta p_b}}{\sqrt{r_\beta p_b}}} > \frac{\frac{\sqrt{r_\alpha p_a} + \sqrt{(1-r_\alpha) p_b}}{\sqrt{r_\alpha p_a}}}{\frac{\sqrt{(1-r_\beta) p_a} + \sqrt{r_\beta p_b}}{\sqrt{(1-r_\beta) p_a}}}$$

which reduces to

$$\frac{\frac{1}{\sqrt{1-r_\alpha}}}{\frac{1}{\sqrt{r_\beta}}} > \frac{\frac{1}{\sqrt{r_\alpha}}}{\frac{1}{\sqrt{1-r_\beta}}}$$

or equivalently

$$\sqrt{r_\alpha r_\beta} > \sqrt{\left(1-r_\alpha\right)\left(1-r_\beta\right)}$$

which clearly holds. ■

We have thus shown that at any p_a and p_b such that the participation threshold equation for signal a, (3), is satisfied, the incentive compatibility condition for signal a is also satisfied.

The argument for incentive compatibility for signal b is analogous. We need to show that if p_a and p_b are such that

$$\frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} > \frac{q(\alpha \mid b)}{q(\beta \mid b)}$$

then it is also the case that

$$\frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} > \frac{q(\alpha \mid b)}{q(\beta \mid b)}$$

So it is enough to show that for all p_a and p_b,

$$\frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} > \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]}$$

which, by Lemma 2, holds when n is large.

To summarize, we have established that

Theorem 1 *In the costly voting model, when n is large, there is an equilibrium in which (i) there is a pair of positive threshold costs; and (ii) voting is sincere.*

Having identified an equilibrium in which voting is sincere, it is useful to examine the efficiency of majority voting in terms of selecting the "correct" candidate—that is, A is elected in state α and B is elected in state β. Notice that sincere voting alone is not enough to guarantee efficiency. Two types of things can still go wrong. First, if too few individuals (in expectation) come to the poll, then even with sincere voting, information will not aggregate. Second, if voters come to the polls in the wrong ratios (based on differences in signal precisions), then again even if there are an infinite number of noters in expectation, information will still not fully aggregate. In the next two sections, we consider each of these potential problems separately. Section 3 examines the limit properties of voter participation decisions whilst Section 4 examines the efficiency of election outcomes.

In the proof of the theorem above we used the assumption that the expected number of voters is large. This enabled us to exploit the approximation formulae for the pivotal probabilities derived by Myerson (1998). As the following example shows, however, a sincere voting equilibrium may exist even when n is small (in fact, we know of no example in which such an equilibrium does not exist).

Example 1 *Suppose that the expected size of the electorate $n = 10$, the signal precisions $r_\alpha = \frac{3}{4}$, $r_\beta = \frac{2}{3}$, and cost distribution $F(c) = \sqrt{2c}$ on $[0, \frac{1}{2}]$. Then the (unique) sincere voting equilibrium has participation probabilities $p_a^* = 0.268$ and $p_b^* = 0.312$.*

The relevant lines depicting the incentive and participation constraints for each type of voter are shown in the figure below.

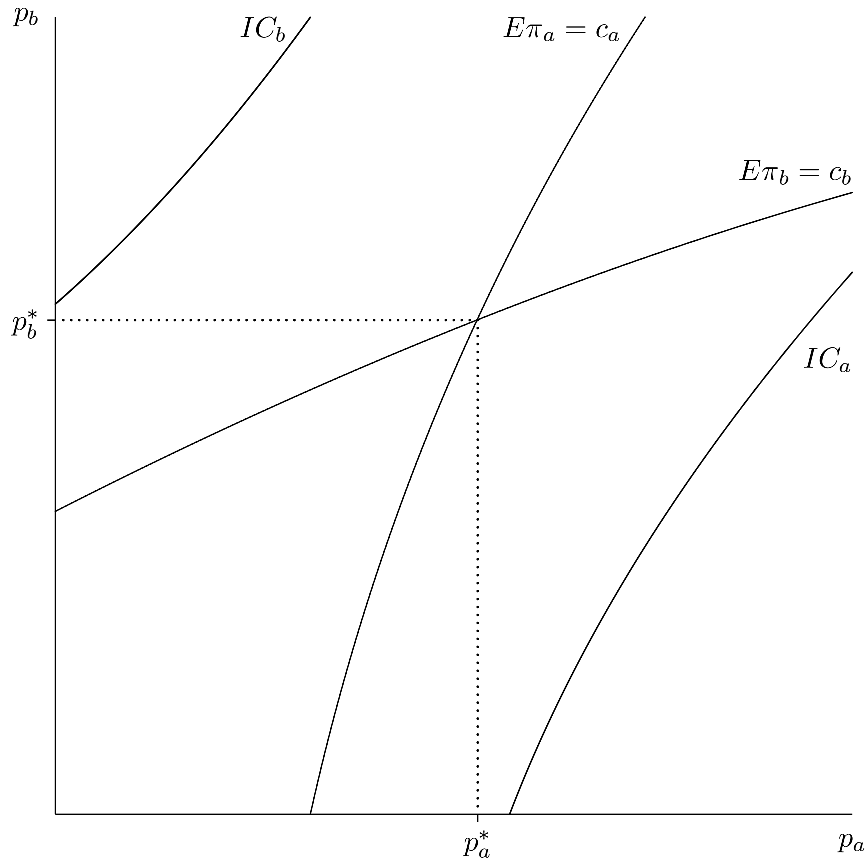

Figure 1: Equilibrium with Sincere Voting

3 Participation in Large Elections

We begin by considering the equilibrium participation thresholds in large elections.

Lemma 3 *In any sequence of sincere voting equilibria, the threshold costs tend to zero; that is,* $\limsup_{n\to\infty} c_a(n) = \limsup_{n\to\infty} c_b(n) = 0$.

Proof. Suppose to the contrary that, for some sequence $\lim_{n\to\infty} c_a(n) > 0$. In that case, the gross benefits (excluding the costs of voting) to voters with a signals from voting must be positive. That is;

$$\lim_{n\to\infty} \left(q(\alpha \mid a) \Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta]\right) > 0$$

where it is understood that the probabilities depend on n.

We know that along the given sequence, $\lim\ p_a(n) > 0$. By taking convergent subsequences, if necessary, suppose that $\lim p_b(n)$ also exists. If $\lim p_b(n) > 0$, then from (7) and (8) we have that $\lim \Pr [Piv_A \mid \alpha] = 0 = \lim \Pr [Piv_A \mid \beta]$. If $\lim p_b = 0$, then

$$\lim \Pr [Piv_A \mid \alpha] = \lim e^{-nr_\alpha p_a} = 0$$

Likewise,

$$\lim \Pr [Piv_A \mid \beta] = \lim e^{-n\left(1-r_\beta\right)p_a} = 0$$

Thus if there is a sequence such that $\lim\ p_a(n) > 0$ then along some subsequence $\lim \Pr [Piv_A \mid \alpha] = 0 = \lim \Pr [Piv_A \mid \beta]$. But this means that along this sequence, the gross benefit of voting for A when the signal is a tends to zero. This contradicts the assumption that $\lim_{n\to\infty} c_a(n) > 0$. ■

Next, we show that, despite the fact that the threshold cost for voting goes to zero in the limit, in any sincere voting equilibrium, the expected number of voters with signal $s \in \{a, b\}$ is unbounded in large elections. There is a "race" between the speed at which the participation thresholds approach zero relative to the size of the electorate. A common intuition one may have is that the "winner" of this race depends on the shape of the cost distribution—particularly in the neighborhood of 0. As we show below, however, sincere voting equilibria have the property that, in large elections, the number of voters becomes unbounded regardless of the shape of this distribution. That is, in the sincere voting model, the problem of too little participation to achieve information aggregation does not arise in the limit.

Lemma 4 *In any sequence of sincere voting equilibria, either* $\lim np_a(n) = \infty$ *or* $\lim np_b(n) = \infty$.

Proof. Suppose to the contrary that $\lim np_a(n) < \infty$ and $\lim np_b(n) < \infty$. In that case, there is a subsequence of threshold probabilities such that the expected number of voters with each signal is finite in the limit. Along such a sequence (or a convergent subsequence, if necessary), it is clear that

$$\lim \left(q(\alpha \mid a) \Pr [Piv_A \mid \alpha] - q(\beta \mid a) \Pr [Piv_A \mid \beta]\right) > 0$$

This, however, contradicts Lemma 3. ■

To establish this property of sincere voting equilibria in large elections, we first prove a technical lemma about the ratio of participation rates in the limit.

Lemma 5 *(i)* $\liminf_{n\to\infty} \frac{p_a(n)}{p_b(n)} > 0$*; and (ii)* $\liminf_{n\to\infty} \frac{p_b(n)}{p_a(n)} > 0$.

Proof. To prove part (i), suppose to the contrary that $\liminf \frac{np_a(n)}{np_b(n)} = 0$. By Lemma 4, it follows that $\liminf np_b(n) = \infty$.

Using the formulae

$$\Pr\left[Piv_B \mid \alpha\right] \approx \frac{e^{n\left(2\sqrt{r_\alpha p_a(1-r_\alpha)p_b}-r_\alpha p_a-(1-r_\alpha)p_b\right)}}{4\sqrt{\pi n\sqrt{r_\alpha p_a\left(1-r_\alpha\right)p_b}}}\frac{\sqrt{r_\alpha p_a}+\sqrt{\left(1-r_\alpha\right)p_b}}{\sqrt{\left(1-r_\alpha\right)p_b}} \quad (11)$$

$$\Pr\left[Piv_B \mid \beta\right] \approx \frac{e^{n\left(2\sqrt{r_\beta p_a(1-r_\beta)p_b}-(1-r_\beta)p_a-r_\beta p_b\right)}}{4\sqrt{\pi n\sqrt{r_\beta p_a\left(1-r_\beta\right)p_b}}}\frac{\sqrt{\left(1-r_\beta\right)p_a}+\sqrt{r_\beta p_b}}{\sqrt{r_\beta p_b}} \quad (12)$$

it may be verified that

$$\frac{\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_B \mid \beta\right]} \approx K_n e^{np_b\left(\left(\sqrt{r_\beta}-\sqrt{(1-r_\beta)\frac{p_a}{p_b}}\right)^2-\left(\sqrt{(1-r_\alpha)}-\sqrt{r_\alpha\frac{p_a}{p_b}}\right)^2\right)}$$

where

$$K_n = \frac{\sqrt{\sqrt{r_\beta\left(1-r_\beta\right)}}}{\sqrt{\sqrt{r_\alpha\left(1-r_\alpha\right)}}}\frac{\left(\sqrt{r_\alpha\frac{p_a}{p_b}}+\sqrt{\left(1-r_\alpha\right)}\right)\sqrt{r_\beta}}{\left(\sqrt{\left(1-r_\beta\right)\frac{p_a}{p_b}}+\sqrt{r_\beta}\right)\sqrt{\left(1-r_\alpha\right)}}$$

and $0 < \lim K_n < \infty$. Taking limits

$$\lim \frac{\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_B \mid \beta\right]} = \infty$$

But this contradicts the individual rationality constraint which requires that

$$q\left(\beta|b\right)\Pr\left[Piv_B \mid \beta\right] - q\left(\alpha|b\right)\Pr\left[Piv_B \mid \alpha\right] \geq 0$$

or

$$\frac{\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_B \mid \beta\right]} \leq \frac{q\left(\beta|b\right)}{q\left(\alpha|b\right)} < \infty$$

The proof of part (ii) is analogous. ■

We are now in a position to show

Lemma 6 *In any sequence of sincere voting equilibria, the expected number of voters tends to infinity; that is,* $\liminf_{n\to\infty} np_a\left(n\right) = \infty = \liminf_{n\to\infty} np_b\left(n\right).$

Proof. The proof follows as a consequence of Lemmas 4 and 5. ■

To summarize, we have shown that, even though the cost thresholds for participation go to zero in the limit in the sincere voting model, they do so sufficiently slowly that the expected number of voters with a and with b signals is unbounded as n gets arbitrarily large.

4 Efficiency in Large Elections

We now turn to the question of whether majority rule is efficient under costly voting. In other words, is it the case that in large elections, the "right" candidate is elected?

Lemma 7 *In any sequence of sincere voting equilibria,*

$$\lim_{n\to\infty} \frac{2\sqrt{r_\alpha p_a (1-r_\alpha) p_b} - r_\alpha p_a - (1-r_\alpha) p_b}{2\sqrt{r_\beta p_a (1-r_\beta) p_b} - (1-r_\beta) p_a - r_\beta p_b} = 1$$

Furthermore,

$$\lim_{n\to\infty} \frac{p_b - p_a}{\sqrt{p_a p_b}} = 2\frac{\sqrt{r_\beta (1-r_\beta)} - \sqrt{r_\alpha (1-r_\alpha)}}{r_\alpha + r_\beta - 1} \tag{13}$$

and hence $p_a \leq p_b$ *for* n *sufficiently large.*

Proof. First, note from Lemma 5 that

$$\liminf \frac{\sqrt{r_\alpha p_a} + \sqrt{(1-r_\alpha) p_b}}{\sqrt{r_\alpha p_a}} < \infty$$

and

$$\liminf \frac{\sqrt{r_\beta p_b} + \sqrt{(1-r_\beta) p_a}}{\sqrt{r_\beta p_b}} < \infty$$

and hence, in the expressions for the pivot probabilities, the exponential terms dominate in the limit. Suppose, contrary to the statement of the Lemma, that the ratio differed from 1 in the limit. In particular, suppose that the ratio converged to a number strictly smaller than 1. In that case

$$\lim \frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} = 0$$

and it would then follow that state β is infinitely more likely in a Piv_A event than is state α.This, however, would imply that the gross benefit to a voter with signal a from voting is negative, which contradicts Lemma 3. Similarly, if the ratio converged to a number strictly greater than 1, then

$$\lim \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} = 0$$

and, it would then follow that state α is infinitely more likely in a Piv_B event than is state β. This, however, would then imply that the gross benefit to a voter with signal b from voting is negative, which also contradicts Lemma 3. Thus the ratio must equal 1 in the limit.

If $r_\alpha = r_\beta$, then $p_a = p_b$. If $r_\alpha > r_\beta > \frac{1}{2}$, rearranging terms in the ratio expression implies

$$\lim \frac{p_b - p_a}{\sqrt{p_a p_b}} = 2\frac{\sqrt{r_\beta (1-r_\beta)} - \sqrt{r_\alpha (1-r_\alpha)}}{r_\alpha + r_\beta - 1}$$

The right-hand side of the above expression is positive. Hence, $p_a < p_b$. ■

We are now in a position to show that information fully aggregates in large elections.

Proposition 2 *In any sequence of sincere voting equilibria, information fully aggregates. Formally, in any sequence of sincere voting equilibria, the probability that candidate A is elected in state α goes to one while the probability that candidate B is elected in state β also goes to one.*

Proof. We will show that when n is large, both

$$r_\alpha p_a - (1 - r_\alpha) p_b > 0 \tag{14}$$

and

$$r_\beta p_b - (1 - r_\beta) p_a > 0 \tag{15}$$

since, together with Lemma 6, this implies a majority of A voters in state α with probability one and a majority of B voters in state β with probability one.

Since $p_a \leq p_b$ from Lemma 7, the required inequality for equation (15) is always satisfied.

We now establish the required inequality for equation (14). Since $r_\beta > \frac{1}{2}$, we have

$$2\frac{\sqrt{r_\beta (1 - r_\beta)} - \sqrt{r_\alpha (1 - r_\alpha)}}{r_\alpha + r_\beta - 1} < 2\frac{\frac{1}{2} - \sqrt{r_\alpha (1 - r_\alpha)}}{r_\alpha - \frac{1}{2}}$$

and so from (13), for large n,

$$\frac{p_b - p_a}{\sqrt{p_a p_b}} < 2\frac{\frac{1}{2} - \sqrt{r_\alpha (1 - r_\alpha)}}{r_\alpha - \frac{1}{2}}$$

or equivalently,

$$\sqrt{\frac{p_b}{p_a}} - \sqrt{\frac{p_a}{p_b}} < 2\frac{\frac{1}{2} - \sqrt{r_\alpha (1 - r_\alpha)}}{r_\alpha - \frac{1}{2}}$$

and since $p_a < p_b$, we have that for all large n

$$\sqrt{\frac{p_b}{p_a}} - 1 < 2\frac{\frac{1}{2} - \sqrt{r_\alpha (1 - r_\alpha)}}{r_\alpha - \frac{1}{2}}$$

Now suppose that the inequality in equation (14) is false; that is, for some large n,

$$r_\alpha p_a - (1 - r_\alpha) p_b \leq 0$$

which is equivalent to

$$\frac{r_\alpha}{1 - r_\alpha} \leq \frac{p_b}{p_a}$$

Substituting this into the inequality derived above we have

$$\sqrt{\frac{r_\alpha}{1-r_\alpha}}-1<2\frac{\frac{1}{2}-\sqrt{r_\alpha\left(1-r_\alpha\right)}}{r_\alpha-\frac{1}{2}}$$

Rearranging this yields

$$\begin{aligned}
\frac{\sqrt{r_\alpha}-\sqrt{1-r_\alpha}}{2\sqrt{1-r_\alpha}} &< \frac{1-2\sqrt{r_\alpha\left(1-r_\alpha\right)}}{2r_\alpha-1} \\
\left(2r_\alpha-1\right)\left(\sqrt{r_\alpha}-\sqrt{1-r_\alpha}\right) &< 2\sqrt{1-r_\alpha}\left(1-2\sqrt{r_\alpha\left(1-r_\alpha\right)}\right) \\
\left(2r_\alpha-1\right)\left(\sqrt{r_\alpha}-\sqrt{1-r_\alpha}\right) &< 2\sqrt{1-r_\alpha}-4\left(1-r_\alpha\right)\sqrt{r_\alpha} \\
\left(3-2r_\alpha\right)\sqrt{r_\alpha} &< \left(3-2r_\alpha\right)\sqrt{1-r_\alpha} \\
\sqrt{r_\alpha} &< \sqrt{1-r_\alpha}
\end{aligned}$$

which is clearly impossible. ■

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