ON THE NATURE OF EQUILIBRIA IN A DOWNSIAN MODEL WITH CANDIDATE VALENCE

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ABSTRACT. I analyze mixed strategy equilibria in a Downsian model with two office-motivated candidates in which one candidate is endowed with a sufficiently large valence advantage that a voter might prefer this candidate even if the voter strictly prefers the other candidate's policies. There is a discrete one-dimensional policy space and the preferences of the median voter are uncertain. I show that there is a range of moderate policies with no gaps that are optimal for the advantaged candidate. There is also a range of liberal policies with no gaps and a corresponding range of conservative policies with no gaps that are optimal actions for the disadvantaged candidate. The upper and lower bounds on these ranges of policies vary in predictable ways with the size of the advantaged candidate's advantage.

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1. INTRODUCTION

One of the most standard frameworks for analyzing candidate policy selection in elections is with a Downsian model (Downs, 1957) in which candidates simultaneously commit to policies on a known policy space and voters vote for whichever candidate proposes the policy they like best. Such a framework fails to take into account that candidates often possess valence characteristics unrelated to policy selection that may be important to voters. For example, a candidate may have greater experience in public office, a superior military background, or previous successes in improving the economy that lead the candidate to be perceived as better equipped to handle problems in the future. These valence characteristics may distinguish a candidate on a 'good-bad' dimension unrelated to whether the candidate chooses a liberal or conservative policy.

This paper presents a characterization of a class of equilibria in a Downsian model in which one candidate possesses a valence advantage. I consider a model in which two office-motivated candidates choose policies simultaneously in a discrete one-dimensional policy space. The precise policy preferences of the median voter are uncertain, but the voter prefers the candidate with superior valence unless the other candidate chooses a policy that is sufficiently closer to the median voter's ideal point. Throughout I restrict attention to cases in which the valence advantage is large enough that a voter might prefer the candidate with superior valence even if the voter strictly prefers the other candidate on the basis of policy positions.

I show that there is a range of moderate policies with no gaps that are optimal for the advantaged candidate. There is also a range of liberal policies with no gaps and a corresponding range of conservative policies with no gaps that are optimal actions for the disadvantaged candidate. The most moderate policy guaranteed to be an optimal action for the disadvantaged candidate is less moderate for larger sizes of the advantaged candidate's advantage. And the most extreme policy that is optimal for the advantaged candidate is more moderate than the most extreme policy that is optimal for the disadvantaged candidate by an amount approximately proportional to the size of the advantaged candidate's advantage.

The analysis of equilibria in a Downsian model with an advantaged candidate differs significantly from that when no candidate has a valence advantage. When there is no candidate valence, the Downsian model predicts that office-motivated candidates will both choose policies equal to the estimated median voter's ideal point. But if one candidate is endowed with a valence advantage, this result no longer holds. In fact, as long as there is uncertainty about voter preferences, pure strategy equilibria normally fail to exist.[1]

To understand why, note that the candidate with a valence advantage wishes to choose the same policy as the disadvantaged candidate so that all voters will strictly prefer the candidate with the valence advantage and the advantaged candidate will win with certainty. Similarly, the disadvantaged candidate wishes to choose a different policy than the advantaged candidate so that there will be at least some chance that the voters prefer the disadvantaged candidate. Since the disadvantaged candidate needs to prevent the advantaged candidate from being able to choose the same policy, this candidate must employ mixed strategies. And to create incentives for the

[1]The only situations in which there are pure strategy equilibria are those in which one candidate's valence advantage is so large that this candidate wins with certainty in equilibrium. Examples of such equilibria can be found in Ansolabehere and Snyder (2000) and Dix and Santore (2002).

disadvantaged candidate to use mixed strategies, the advantaged candidate must also use mixed strategies. This necessitates the analysis of mixed strategy equilibria.

To the best of my knowledge there has only been one attempt to characterize mixed strategy equilibria in a Downsian model with candidate valence. Aragones and Palfrey (2002) consider an analogous model to that in the present paper with two office-motivated candidates competing in a discrete one-dimensional policy space with uncertainty about the preferences of the median voter. However, Aragones and Palfrey (2002) focus attention on cases in which candidate valence is minimal in the sense that voters vote for the candidate with superior valence if they are indifferent between the policies proposed by the two candidates, but vote for whichever candidate proposes the voter's preferred policy otherwise.

When candidate valence is minimal, Aragones and Palfrey (2002) note that there is a mixed strategy equilibrium in which both candidates randomize amongst a small number of moderate policies. The advantaged candidate chooses slightly more moderate policies on average, and in the limit as the number of points in the policy space becomes large, the candidates choose strategies arbitrarily close to those they would use if there were no candidate valence. However, Aragones and Palfrey (2002) also note that this type of equilibrium might fail to exist if candidate valence is sufficiently large that a voter might prefer the candidate with superior valence even if the voter prefers the other candidate on the basis of policy positions. I consider the form of equilibria when a candidate's valence advantage is large in this paper.

Though I do not characterize all equilibria in this game, my paper does give a characterization of the optimal policy choices for the candidates that may arise in a particular equilibrium of the game. And while I am able to derive which policies will be optimal in equilibrium, it should be noted that I do not calculate the precise probabilities with which the candidates take various actions. Instead I use indirect arguments to show that there will be a mixed equilibrium of a particular form. The proof techniques used in this paper are thus entirely different from those in Aragones and Palfrey (2002).

Other papers on candidate valence address different issues than finding mixed equilibria in a candidate location game when two office-motivated candidates choose policies simultaneously. Aragones and Palfrey (2004) show that the predictions in Aragones and Palfrey (2002) are supported in an experimental setting. Aragones and Palfrey (2005) and Groseclose (2001) analyze equilibria in a candidate location game in which the candidates have policy motivations. Adams (1999) and Schofield (2004) consider the effect of candidate valence on equilibria when there are multiple candidates. Berger *et al.* (2000) and Bernhardt and Ingberman (1985) consider models in which policy choices are not known with certainty but some candidate has the advantage of a lower variance in policy action. Ansolabehere *et al.* (2001), Fiorina (1973), and Stone and Simas (2007) conduct empirical studies on the effect of incumbency advantages on candidate policy selection. Finally, several papers analyze valence effects in the context of electoral contests in which campaign expenditures or other costly actions can affect the outcome of the election (e.g. Ashworth and Bueno de Mesquita, 2009; Carillo and Castanheria, 2008; Erikson and Palfrey, 2000; Iaryczower and Mattozzi, 2009; Meirowitz, 2008; Prat, 2002a; b).

While mixed strategies might seem like a slightly unnatural way for candidates to choose policies, it is worth noting that it is fairly standard to analyze mixed strategy equilibria in such games. In addition to work on candidate valence, several papers have considered mixed strategy equilibria in the context of elections in which candidates choose policies in multi-dimensional spaces (e.g. Banks *et al.*, 2002; Duggan, 2007; Duggan and Jackson, 2005; McKelvey, 1986). And researchers have also analyzed mixed strategy equilibria in Hotelling's (1929) related game in which firms can compete by choosing locations (e.g. Bester, 1996; Osborne and Pitchik, 1986; 1987; Shaked, 1982).

2. The Model

I study a model very similar to that in Aragones and Palfrey (2002). There is an election between two candidates A and D, where A is the candidate with superior valence. The candidates choose policies from the policy space $X = \{x_1, \ldots, x_n\}$, where $x_i = \frac{i-1}{n-1}$ for all $i = 1, \ldots, n$. I let x_A denote the policy chosen by A and x_D the policy chosen by D. Each candidate obtains a payoff equal to the candidate's probability of winning the election.

The median voter has an ideal point $x_m \in X$. This ideal point is not observed by either candidate. Instead, each candidate shares a common belief that the probability the median voter's ideal point is x_i is ρ_i, where $\rho_i \geq 0$ for all i and $\sum_{i=1}^{n} \rho_i = 1$. If A wins the election, then the median voter obtains a utility of $U_m(x_A) = \delta - |x_m - x_A|$, where $\delta > 0$. If D wins the election, then the median voter obtains a utility of $U_m(x_D) = -|x_m - x_D|$.

I assume, as do Aragones and Palfrey (2002), that n is even and $\rho_i = \frac{1}{n}$ for all i. However, I also note throughout the manuscript how the results would be affected by the more general assumption that the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space or $\rho_i = \rho_{n-i+1}$ for all i and ρ_i is nondecreasing in i for all $i \leq \frac{n}{2}$. Most of the results continue to hold with virtually identical proof techniques as long as the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space.

I also assume that there is some positive integer a for which $\frac{a-1}{n-1} < \delta < \frac{a}{n-1}$. This assumption will hold as long as δ is not an integral multiple of $\frac{1}{n-1}$ and is solely made to ensure that the median voter will always have strict preferences between the two candidates. The only difference between the model I consider and that in Aragones and Palfrey (2002) is that Aragones and Palfrey (2002) restrict attention to the case $a = 1$, but I assume throughout that $1 < a < \frac{n}{2}$. The assumption that $a > 1$ means that a voter might prefer candidate A even if the voter strictly prefers D's policy position. The assumption that $a < \frac{n}{2}$ is made to ensure that δ is small enough that the candidate with superior valence will not win with certainty in equilibrium.

The game proceeds as follows. Both candidates simultaneously choose policy positions from the policy space X. The median voter observes these policy choices and votes for whichever candidate affords the voter a higher utility. The candidate chosen by the median voter is then elected. A justification for restricting attention to the median voter in this setting can be found in Groseclose (2007).

A strategy for a candidate $\sigma = (\sigma_1, \ldots, \sigma_n)$ specifies the probability σ_i with which the candidate chooses each policy x_i in the policy space X. I let σ^A denote the strategy chosen by A and σ^D the strategy chosen by D. Throughout the paper I also let x_i denote the strategy in which a candidate chooses x_i with probability 1.

If candidate A uses the strategy σ^A and candidate D uses the strategy σ^D, I let $\Pi_A(\sigma^A, \sigma^D)$ denote the probability that A wins the election and let $\Pi_D(\sigma^A, \sigma^D) = 1 - \Pi_A(\sigma^A, \sigma^D)$ denote the probability that D wins the election. An equilibrium is a profile of strategies (σ^A, σ^D) such that $\Pi_A(\sigma^A, \sigma^D) \geq \Pi_A(\sigma^{A'}, \sigma^D)$ for all other strategies $\sigma^{A'}$ and $\Pi_D(\sigma^A, \sigma^D) \geq \Pi_D(\sigma^A, \sigma^{D'})$ for all other strategies $\sigma^{D'}$. I also let $\pi_A(x_A, x_D)$ denote the probability that A wins the election if A and D

choose policies x_A and x_D respectively and let $\pi_D(x_A, x_D) = 1 - \pi_A(x_A, x_D)$ denote the corresponding probability that D wins the election after these policy choices.

3. Preliminaries

The mixed strategy equilibrium in this game need not be unique, and I do not attempt to characterize all equilibria. Instead, I seek to demonstrate that there is an equilibrium which satisfies certain properties. In particular, I seek to show that there is a range of moderate policies with no gaps that are optimal for the advantaged candidate. I also seek to show that there is a range of liberal policies with no gaps and a corresponding range of conservative policies with no gaps that are optimal actions for the disadvantaged candidate. Finally I wish to characterize the bounds on these ranges of optimal actions. First I derive some preliminary results.

Note that the game described in the previous section is a constant sum game since the sum of the candidates' payoffs is always equal to 1. We thus know from von Neumann (1928) that a strategy profile is an equilibrium if and only if each player's strategy is a maxminimizer strategy, or a strategy which maximizes the minimum payoff the player can guarantee himself without knowing what strategy the opponent will use. Formally, we have the following result:

Theorem 0. *$(\sigma^{A^*}, \sigma^{D^*})$ is an equilibrium if and only if σ^{A^*} and σ^{D^*} are maxminimizer strategies. That is, $(\sigma^{A^*}, \sigma^{D^*})$ is an equilibrium if and only if σ^{A^*} maximizes $\min_{\sigma^D} \Pi_A(\sigma^A, \sigma^D)$ with respect to σ^A and σ^{D^*} maximizes $\min_{\sigma^A} \Pi_D(\sigma^A, \sigma^D)$ with respect to σ^D. This is equivalent to σ^{A^*} minimizing $\max_{\sigma^D} \Pi_D(\sigma^A, \sigma^D)$ and σ^{D^*} minimizing $\max_{\sigma^A} \Pi_A(\sigma^A, \sigma^D)$.*

In the present game, a strategy is a maxminimizer strategy if and only if it is a minmaximizer strategy, or a strategy which minimizes the maximum payoff the other player can obtain. It is also worth noting that all equilibria in this game are payoff equivalent. In any equilibrium, candidate A obtains payoff $\overline{\Pi}_A \equiv \max_{\sigma^A} \min_{\sigma^D} \Pi_A(\sigma^A, \sigma^D)$ and candidate D obtains payoff $\overline{\Pi}_D \equiv \max_{\sigma^D} \min_{\sigma^A} \Pi_D(\sigma^A, \sigma^D)$. Furthermore, equilibrium strategies are interchangeable, and a strategy is an equilibrium strategy if and only if it guarantees the candidate will receive at least her equilibrium payoff. Thus $(\sigma^{A^*}, \sigma^{D^*})$ is an equilibrium if and only if $\Pi_A(\sigma^{A^*}, x_k) \geq \overline{\Pi}_A$ for all $x_k \in X$ and $\Pi_D(x_i, \sigma^{D^*}) \geq \overline{\Pi}_D$ for all $x_i \in X$.

These results also indicate that the set of equilibria is convex. If σ' and σ'' are maxminimizer strategies for candidate A, then $\Pi_A(\sigma', x_k) \geq \overline{\Pi}_A$ and $\Pi_A(\sigma'', x_k) \geq \overline{\Pi}_A$ for all $x_k \in X$, $\Pi_A(\alpha\sigma' + (1-\alpha)\sigma'', x_k) = \alpha\Pi_A(\sigma', x_k) + (1-\alpha)\Pi_A(\sigma'', x_k) \geq \overline{\Pi}_A$ for all $\alpha \in (0,1)$ and $x_k \in X$, and $\alpha\sigma' + (1-\alpha)\sigma''$ is also a maxminimizer strategy for candidate A. Similarly, the set of maxminimizer strategies for candidate D is convex, and the set of equilibria is convex.

I now derive expressions for the probabilities with which the various candidates win the election as a function of the candidate policy choices. Suppose A chooses the policy x_i and D chooses the policy x_k. Note that if $k \in [i-a+1, i+a-1]$, then any voter strictly prefers candidate A regardless of the voter's ideal point. If $k = i - a - 2j$ for some integer $j \geq 0$ or $k = i - a - 2j + 1$ for some integer $j \geq 1$, then voters with ideal points no greater than $i - a - j$ prefer candidate D and voters with ideal points greater than $i - a - j$ prefer candidate A. Similarly, if $k = i + a + 2j$ for some integer $j \geq 0$ or $k = i + a + 2j - 1$ for some integer $j \geq 1$, then voters with ideal points greater than or equal to $i + a + j$ prefer candidate D and voters with ideal points less than $i + a + j$ prefer candidate A. From this I obtain the following expression for $\pi_A(x_A, x_D)$:[2]

$$\pi_A(x_i, x_k) = \begin{cases} \frac{n-i+a+j}{n}, & k = i - a - 2j, j \in \mathbb{Z}^* \text{ or } k = i - a - 2j + 1, j \in \mathbb{Z}^+ \\ 1, & k \in [i - a + 1, i + a - 1] \\ \frac{i+a+j-1}{n}, & k = i + a + 2j, j \in \mathbb{Z}^* \text{ or } k = i + a + 2j - 1, j \in \mathbb{Z}^+ \end{cases} \tag{1}$$

From this expression for $\pi_A(x_A, x_D)$, we see that there is a natural symmetry about the center of the policy space X in the sense that $\pi_A(x_i, x_k) = \pi_A(x_{n-i+1}, x_{n-k+1})$ for all i and k. Given this symmetry, it seems natural to expect that candidates might employ symmetric strategies in which they select the policies x_i and x_{n-i+1} with the same probability. Formally, I define a symmetric strategy as follows:

Definition. *A strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ is symmetric if and only if $\sigma_i = \sigma_{n-i+1}$ for all i. Σ is the set of symmetric strategies available to the candidates.*

I first note that there is indeed an equilibrium in symmetric strategies:

Theorem 1. *There is an equilibrium in which both candidates use symmetric strategies.*

All proofs are in the appendix. In proving Theorem 1, I show that if a candidate is using an asymmetric strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$, then the candidate can do at least as well by using

[2] Throughout this expression I let $\mathbb{Z}^*$ denote the set of nonnegative integers and $\mathbb{Z}^+$ denote the set of positive integers.

the symmetric strategy $\sigma' = (\sigma'_1, \ldots, \sigma'_n)$ in which $\sigma'_i = \frac{\sigma_i + \sigma_{n-i+1}}{2}$ for all i. In particular, if σ is a maxminimizer strategy, then the corresponding symmetric strategy σ' must also be a maxminimizer strategy. This result holds as long as the distribution of the median voter's ideal point is symmetric about the center of the policy space.

Since there are equilibria in symmetric strategies, I focus attention on characterizing the properties of symmetric equilibria throughout the remainder of the paper. I first introduce a few definitions.

Definition. *Σ^A is the set of symmetric maxminimizer strategies for candidate A.*

Since the set of maxminimizer strategies is equivalent to the set of strategies a candidate may use in equilibrium, Σ^A is also the set of symmetric strategies that A may use in equilibrium. We thus know from Theorem 1 that this set of strategies is nonempty. I now introduce a particular kind of symmetric strategy.

Definition. *For any integer i satisfying $1 \leq i \leq \frac{n}{2}$, σ^i is the strategy in which a candidate chooses x_i with probability $\frac{1}{2}$ and x_{n-i+1} with probability $\frac{1}{2}$. That is, $\sigma^i = (\sigma^i_1, \ldots, \sigma^i_n)$ is the vector of probabilities satisfying $\sigma^i_i = \sigma^i_{n-i+1} = \frac{1}{2}$ and $\sigma^i_k = 0$ for all k such that $k \neq i$ and $k \neq n - i + 1$.*

σ^i is the simplest kind of symmetric strategy that a candidate may use. It involves a candidate mixing between exactly two actions on opposite sides of the policy space X. This particular type of symmetric strategy is useful because any other symmetric strategy can be expressed as a combination of several of these simpler symmetric strategies. In particular, if $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a symmetric strategy, then σ can be written as $\sigma = \sum_{i=1}^{n/2} 2\sigma_i\sigma^i$. This expresses the symmetric strategy σ in terms of how much weight σ places on each of the strategies of the form σ^i with $1 \leq i \leq \frac{n}{2}$.

In analyzing the properties of symmetric equilibria, it is helpful to figure out how putting more or less weight on the various strategies σ^i with $1 \leq i \leq \frac{n}{2}$ affects candidate A's payoff. In particular, I consider how A's payoff from choosing each of the strategies of the form σ^i varies with i in response to a given pure strategy policy choice by D. This is done in the following lemma:

Lemma 1. *Suppose D chooses an action x_k with $k \leq \frac{n}{2}$. Then we have the following:*

(a). All strategies of the form σ^i for any integer i satisfying $k + a \leq i \leq \frac{n}{2}$ afford the same expected payoff for A against D's action.

(b). For any positive integer $i \leq k - a + 1$, A's expected payoff from using the strategy σ^i is strictly increasing in i.

(c). For any positive integer i satisfying $i \leq k+a-1$ and $i \leq \frac{n}{2}$, A's expected payoff from using the strategy σ^i is nondecreasing in i.

This result guarantees that if D uses an action x_k with $k \leq \frac{n}{2}$, then A's payoff from using the strategy σ^i is at least as large as A's payoff from using the strategy σ^{i-1} as long as $k \neq i-a$. It is only in the case where $k = i-a$ that A might benefit from choosing the strategy σ^{i-1} instead of σ^i. This insight also holds as long as the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space.

To understand the intuition behind this result, consider what action A would like to take if D chooses the action x_k for some $k \leq \frac{n}{2}$. The best scenario for A would be to choose an action x_i for some i satisfying $k-a+1 \leq i \leq k+a-1$, as this would ensure that A wins the election with probability 1. However, if A is not choosing an action which wins with probability 1 against D's choice of x_k, then A would like to at least choose an action that is as close to x_k as possible, as this would maximize A's chances of winning amongst actions that do not win with probability 1. For example, if A chooses the policy x_{k+a+1}, then A wins if the median voter has an ideal point greater than or equal to x_{k+1}, but if A chooses the policy x_{k+a+2}, then A only wins if the median voter has an ideal point greater than or equal to x_{k+2}.

The fact that A benefits from choosing actions that are closer to D's choice of action makes the results in Lemma 1 intuitive. Suppose A uses a strategy of the form σ^i for some i satisfying $k+a \leq i < \frac{n}{2}$, and consider what happens when A instead uses the strategy σ^{i+1}. This change from σ^i to σ^{i+1} means that A now mixes between x_{i+1} and x_{n-i} instead of mixing between x_i and x_{n-i+1}. x_{i+1} is further away from x_k than x_i, so changing from x_i to x_{i+1} hurts A's chances of winning. However, x_{n-i} is closer to x_k than x_{n-i+1}, so changing from x_{n-i+1} to x_{n-i} improves A's chances of winning. The benefits from changing from x_{n-i+1} to x_{n-i} precisely cancel out the losses from changing from x_i to x_{i+1}, so A is indifferent between the strategies σ^i and σ^{i+1}. This gives the result in part (a).

Now suppose A uses a strategy of the form σ^i for some i satisfying $i < k-a+1$, and consider what happens when A instead uses the strategy σ^{i+1}. In this case, x_{i+1} is closer to x_k than x_i and x_{n-i} is closer to x_k than x_{n-i+1}. Thus A is strictly better off mixing between x_{i+1} and x_{n-i} than mixing between x_i and x_{n-i+1}, and A strictly prefers using the strategy σ^{i+1} to using the strategy σ^i. This gives the result in part (b).

Finally, suppose A uses a strategy of the form σ^i for some i satisfying $k-a+1 \leq i < k+a-1$ and $i \leq \frac{n}{2}$, and consider what happens when A instead uses the strategy σ^{i+1}. In this case, x_{i+1} might not be closer to x_k than x_i, but if A chooses x_{i+1}, then A wins with probability 1. And x_{n-i} is definitely closer to x_k than x_{n-i+1}. Thus A cannot lose utility either from switching to x_{i+1} from x_i or from switching to x_{n-i} from x_{n-i+1}. Thus A does at least as well by using the strategy σ^{i+1} as by using the strategy σ^i, and we have the result in part (c).

I now consider how D's payoff is affected from various changes in D's action if A uses one of these strategies σ^i for some $i \leq \frac{n}{2}$.

Lemma 2. *Suppose A uses the strategy σ^i for some $i \leq \frac{n}{2}$. Then we have the following:*

(a). For any positive integer $k \leq i-a$, D's expected payoff from choosing the action x_k is strictly increasing in k.

(b). All actions of the form x_k for positive integers k satisfying $i+a \leq k \leq n-i+1-a$ afford D the same expected payoff against A's strategy.

(c). For any positive integer k satisfying $i-a+1 \leq k \leq n-i+1-a$, D's expected payoff from choosing the action x_k is nondecreasing in k.

(d). For any positive integer k satisfying $i-a+1 \leq k \leq i+a-1$ and $k+2 \leq n-i+1-a$, D's expected payoff from choosing the action x_{k+2} is strictly greater than D's expected payoff from choosing the action x_k.

Lemma 2 indicates that if A uses some symmetric strategy σ^i with $i \leq \frac{n}{2}$ and $k+1 \leq n-i+1-a$, then D's payoff from using the action x_{k+1} will be at least as high as D's payoff from using the action x_k as long as $i \neq k+a$. It is only when $i = k+a$, that D might do better by choosing the policy x_k instead of x_{k+1}. Similarly, D's payoff from using an action x_{k+1} with $k+1 \leq \frac{n}{2}$ and $k+1 \leq n-i+1-a$ is at least as high as D's payoff from using the action x_k when $i \neq k+a$ if the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space.

To understand the intuition behind this result, first consider what action D would like to take if A took the action x_i. D would want to avoid choosing an action x_k with $i-a+1 \leq k \leq i+a-1$, as D would then lose with certainty. However, D would like to choose an action as close to x_i as possible without choosing an action so close that D loses with certainty. For example, if D chooses the action x_{i-a}, then D wins if the median voter's ideal point is no greater than x_{i-a}, but if D

chooses the action x_{i-a-1}, then D only wins if the median voter's ideal point is no greater than x_{i-a-1}.

The fact that D would like to choose an action as close to A's action as possible without losing with certainty can help one understand the results in Lemma 2. If A uses the strategy σ^i for some $i \leq \frac{n}{2}$, then A is mixing between x_i and x_{n-i+1}. Thus if D chooses an action x_k with $k \leq i - a$, larger values of k mean that D chooses an action that is closer to both x_i and x_{n-i+1}, but not so close that D would lose with certainty against either of these actions. Thus if D chooses an action x_k with $k \leq i - a$, D's expected payoff is increasing in k. This gives the result in part (a).

If D chooses an action x_k with $i + a \leq k \leq n - i + 1 - a$, then choosing a larger value of k means that D chooses an action that is further from x_i but closer to x_{n-i+1} while still being sufficiently far from both x_i and x_{n-i+1} that D would not lose with certainty against either of these actions. The benefits from being closer to x_{n-i+1} precisely cancel out the losses from being further from x_i, so D is indifferent between all actions of this form, as indicated in part (b).

Finally, suppose D chooses an action x_k with $i - a + 1 \leq k \leq i + a - 1$ and $k < n - i + 1 - a$, and consider what happens when D takes the action x_{k+1} instead of x_k. In this case, x_{k+1} may very well be further from x_i than x_k. However, x_k loses with certainty for D if A chooses x_i, so x_{k+1} cannot do any worse for D against x_i than x_k. And x_{k+1} is closer to x_{n-i+1} than x_k without being so close that D would lose with certainty if D chose the action x_{k+1} and A chose the action x_{n-i+1}. Thus x_{k+1} does at least as well for D as x_k if A chooses either of the actions x_i or x_{n-i+1}. Furthermore, if $k + 2 \leq n - i + 1 - a$, then D does strictly better by using the action x_{k+2} instead of x_k when A chooses the action x_{n-i+1}, and D strictly prefers x_{k+2} to x_k when A is using the strategy σ^i. This gives the results in parts (c) and (d).

Before proceeding to the main results, I first note one additional property of A's maxminimizer strategies.

Lemma 3. *There is an equilibrium in which A uses a symmetric strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that $\sigma_i = 0$ for all positive integers i satisfying $1 \leq i \leq a - 1$.*

The reasoning behind this lemma is as follows. As noted previously, A would like to choose an action as close to D's action as possible. If A chooses the action x_a instead of an action of the form x_i with $i \leq a - 1$, then A's policy will be closer to D's policy unless D chooses a policy x_k with $k \leq a - 1$. However, if D chooses a policy x_k with $k \leq a - 1$, then A wins with certainty if A uses the policy x_a. Thus A can only benefit from choosing the action x_a instead of choosing an action of the form x_i with $i \leq a-1$, and there is no need for A to ever choose any of the actions x_i with $i \leq a-1$ with positive probability. Similarly, there is no need for A to ever choose any of the $a-1$ actions closest to x_n. This result also holds as long as the distribution of the median voter's ideal point is symmetric about the center of the policy space.

Given this result, I focus on equilibria in which A does not use the $a-1$ actions closest to either of the endpoints of the policy space for the remainder of the paper. I give one more definition to describe this set of strategies.

Definition. *Σ_a is the set of $\sigma = (\sigma_1, \ldots, \sigma_n)$ in Σ for which $\sigma_i = 0$ for all positive integers i satisfying $1 \leq i \leq a-1$. Σ_a^A is the set of $\sigma = (\sigma_1, \ldots, \sigma_n)$ in Σ^A for which $\sigma_i = 0$ for all positive integers i satisfying $1 \leq i \leq a-1$.*

Thus Σ_a is the set of symmetric strategies in which a candidate does not use the $a-1$ actions closest to the endpoints of the policy space, and Σ_a^A is the set of such strategies that A may use in equilibrium. Since we know from Lemma 3 that Σ_a^A is nonempty, I focus on characterizing the properties of strategies in this set.

4. Main Results

This section characterizes the optimal actions for the candidates in a particular equilibrium. I ultimately wish to show that there is a range of moderate policies with no gaps that are optimal for the candidate with superior valence. I also wish to show that there is a range of liberal policies with no gaps and a range of conservative policies with no gaps that are optimal for the lower quality candidate. A range of policies with no gaps simply refers to a range of policies such that, if any two policies are in that range, all policies in between these policies are also in that range. Formally, I define gaps as follows.

Definition. *A strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ has a gap at x_i if $\sigma_i = 0$ and there exist positive integers j and k such that $j < i < k$, $\sigma_j > 0$, and $\sigma_k > 0$.*

Definition. *A strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ has no gaps if σ does not have a gap at x_i for all positive integers i satisfying $1 \leq i \leq n$.*

Strategies with gaps are significant because if the advantaged candidate uses a strategy with a gap, then this gives valuable information about what actions will not be optimal for the disadvantaged candidate. In particular, I prove the following lemma:

Lemma 4. *Suppose A uses a symmetric strategy σ that has a gap at x_i for some positive integer i satisfying $a+1 \leq i \leq \frac{n}{2}$. Also suppose that $\sigma_{i-1} > 0$ for this i. Then it is not a best response for D to take the action x_{i-a}.*

This lemma guarantees that if A is using a symmetric strategy σ with a gap at x_i but not at x_{i-1} for some positive integer i satisfying $a+1 \leq i \leq \frac{n}{2}$, then x_{i-a} is not a best response for D. This result follows by repeatedly using the results in Lemma 2. I have indicated in my discussion of Lemma 2 that if A is using some symmetric strategy σ^j with $j \leq \frac{n}{2}$, then the only way D might be strictly better off by taking the action x_{i-a} instead of x_{i-a+1} is if $j = i$. Against all other symmetric strategies σ^j with $j \neq i$, D does at least as well by taking the action x_{i-a+1} instead of x_{i-a}.

Now if A is using a symmetric strategy that has a gap at x_i, then A is putting no weight on the strategy σ^i. A is only using symmetric strategies σ^j with $j \neq i$. But since D is at least as well off against all of these strategies by taking the action x_{i-a+1} instead of x_{i-a}, it is intuitive that D would not want to take the action x_{i-a}. The technical condition that $\sigma_{i-1} > 0$ enables me to show that x_{i-a} is in fact strictly suboptimal for D. This result also holds with virtually identical reasoning as long as the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space.

Lemma 4 can be used to show that there is an equilibrium in which A employs a symmetric strategy with no gaps:

Theorem 2. *There is a strategy in Σ_a^A with no gaps. Furthermore, if there is a strategy $\sigma \in \Sigma_a^A$ such that $\sigma_j > 0$ and $\sigma_i = 0$ for all $i < j$ for some $j \leq \frac{n}{2}$, then for this j there is also a strategy $\sigma' \in \Sigma_a^A$ with no gaps such that $\sigma'_j > 0$ and $\sigma'_i = 0$ for all $i < j$.*

To see that A has a maxminimizer strategy with no gaps, note that if A is using a strategy in Σ_a with a gap at x_i but not at x_{i-1} for some $i \leq \frac{n}{2}$, then we know from Lemma 4 that x_{i-a} is not a best response for D to A's choice of strategy. Thus if A uses such a strategy, D would take an action x_k with $k \neq i-a$. Now I have indicated in my discussion of Lemma 1 that if D takes an action x_k with $k \neq i-a$, then A's payoff from the strategy σ^i is at least as high as A's payoff from the strategy σ^{i-1}. This means that if A puts slightly more weight on the strategy σ^i and slightly less weight on the strategy σ^{i-1}, then A does at least as well against any action which D would consider taking against A's original strategy. But if A makes this change, then there is no longer a gap in A's strategy at x_i. Thus A has a symmetric maxminimizer strategy with no gaps.

The second part of the theorem simply shows that A can still use the most extreme policy that may be optimal when A is restricted to employing a strategy with no gaps. Again Theorem 2 holds with virtually identical reasoning as long as the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space.

The fact that A has a symmetric maxminimizer strategy with no gaps enables one to show that in every equilibrium, there is a range of actions with no gaps around the center of the policy space which must be best responses for A to D's strategy. This is illustrated in Theorem 3.

Theorem 3. *Let j be the smallest positive integer such that there is some $\sigma \in \Sigma_a^A$ with $\sigma_j > 0$. Then in every equilibrium (σ^A, σ^D), all actions of the form x_i with $j \leq i \leq n - j + 1$ must be best responses for A to D's strategy.*

This result is an immediate consequence of Theorem 2. If j is defined as above, then we know from Theorem 2 that there is a symmetric maxminimizer strategy with no gaps for A, $\sigma^{A'}$, such that $\sigma_j^{A'} > 0$. Also, if (σ^A, σ^D) is an equilibrium, then σ^D is a maxminimizer strategy for D. And since $\sigma^{A'}$ and σ^D are both maxminimizer strategies, $(\sigma^{A'}, \sigma^D)$ must be an equilibrium. But $\sigma^{A'}$ takes all actions of the form x_i with $j \leq i \leq n-j+1$ with positive probability. Since $(\sigma^{A'}, \sigma^D)$ is an equilibrium, these actions must all be best responses for A to D's strategy, and the result follows.

While I am focusing on symmetric equilibria in this paper, it is worth noting that this result also holds for every equilibrium in which some player does not use a symmetric strategy. And Theorem 3 also holds with identical reasoning as long as the distribution of the median voter's ideal point is weakly single-peaked and symmetric about the center of the policy space.

Though there will typically be gaps in D's strategy near the center of the policy space, I illustrate in the next two theorems that there is an equilibrium in which D has a range of actions with no gaps that are all best responses to A's strategy. This range of actions corresponds to some set of policies that are at least as liberal as $x_{n/2-a}$. First I prove the following:

Theorem 4. *Suppose there is a strategy in Σ_a^A such that one of D's best responses to this strategy is x_j for some $j < \frac{n}{2} - a$. Then there is also a strategy in Σ_a^A such that all actions of the form x_k with $j \leq k \leq \frac{n}{2} - a$ are best responses for D.*

This theorem does not guarantee that there will be an equilibrium in which D takes all the actions x_k in the range $j \leq k \leq \frac{n}{2} - a$ with positive probability, as D need not choose to take all optimal actions with positive probability in equilibrium. However, it does ensure that these actions will all be optimal for D in equilibrium. The symmetry of the problem then guarantees that all conservative policies of the form x_k with $\frac{n}{2} + a + 1 \leq k \leq n - j + 1$ are also best responses for D.

The reasoning behind Theorem 4 is similar to the reasoning behind Theorem 2. Suppose A is using a symmetric strategy such that x_{i-a} is not a best response for D for some $i \leq \frac{n}{2}$ but x_{i-a-1} is a best response for D. Similar reasoning to that in Lemma 4 shows that if this holds, then A must be putting positive weight on the strategy σ^{i-1}.

Since A is putting positive weight on the strategy σ^{i-1}, A can put slightly more weight on the strategy σ^i and slightly less weight on the strategy σ^{i-1}. As noted in the reasoning behind Theorem 2, such a change is favorable to A if D is using actions of the form x_k with $k \neq i - a$. Thus A prefers to keep putting more weight on the strategy σ^i and less weight on the strategy σ^{i-1} until x_{i-a} is also a best response for D. This gives the result in Theorem 4.

While Theorem 4 illustrates that there is an equilibrium in which all actions of the form x_k with $j \leq k \leq \frac{n}{2} - a$ are best responses for D if there is an equilibrium in which x_j is a best response for D, it does not guarantee that there is an equilibrium in which x_j is a best response for D for some $j \leq \frac{n}{2} - a$. For instance, this theorem leaves open the possibility that all of D's best responses will be near the center of the policy space and be of the form x_k with $\frac{n}{2} - a + 1 \leq k \leq \frac{n}{2} + a$. This possibility is ruled out in Theorem 5.

Theorem 5. *If $\sigma \in \Sigma_a^A$, then one of D's best responses to σ is to choose an action of the form x_k for some positive integer $k \leq \frac{n}{2} - a$.*

To understand the intuition behind this result, suppose A were using a symmetric strategy, but D did not have a best response of the form x_k for some positive integer $k \leq \frac{n}{2} - a$. Since A is using a symmetric strategy, D's expected payoff from using the action x_k is the same as D's expected payoff from using the action x_{n-k+1}, and D also does not have a best response of the form x_k with

$k \geq \frac{n}{2} + a + 1$. This means all of D's best responses are of the form x_k for some k satisfying $\frac{n}{2} - a + 1 \leq k \leq \frac{n}{2} + a$, the policies closest to the center of the policy space.

But if D is using an action of this form, then A can improve his payoff by putting relatively more weight on the strategy $\sigma^{n/2}$ and putting relatively less weight on the other strategies σ^i for $i < \frac{n}{2}$ that A was using before. The strategy $\sigma^{n/2}$ which mixes between $x_{n/2}$ and $x_{n/2+1}$ wins with probability one if D is using an action of the form x_k with $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$. And if D is using the actions $x_{n/2-a+1}$ or $x_{n/2+a}$, then one of the actions $x_{n/2}$ or $x_{n/2+1}$ will with probability one, and the other action will be as close to D's action as possible without being close enough to win with certainty.

No other symmetric strategy σ^i is more effective for A when D is using actions of the form x_k with $\frac{n}{2} - a + 1 \leq k \leq \frac{n}{2} + a$ than $\sigma^{n/2}$, so this means that A can improve his or her payoff by putting relatively more weight on the strategy $\sigma^{n/2}$ and relatively less weight on the other strategies. Thus if A is using a symmetric maxminimizer strategy, one of D's best responses must be of the form x_k with $k \leq \frac{n}{2} - a$.

While the results for the advantaged candidate in Theorems 2 and 3 hold for any distribution of the median voter's ideal point that is weakly single-peaked and symmetric about the center of the policy space, the results for the disadvantaged candidate in Theorems 4 and 5 will not necessarily hold under this more general treatment. Thus the assumption that the distribution of the median voter's ideal point is uniform is useful in deriving properties of the disadvantaged candidate's optimal actions.

By combining Theorems 1-5, I obtain the following result:

Theorem 6. *There is an equilibrium (σ^A, σ^D) in symmetric strategies characterized by two positive integers k_A and k_D satisfying $k_D \leq \frac{n}{2} - a$ and $a \leq k_A \leq \frac{n}{2}$ such that the following hold:*

(a). All actions of the form x_i with $k_A \leq i \leq n - k_A + 1$ are best responses for A to D's strategy.

(b). $\sigma_i^A = 0$ if $i < k_A$ or $i > n - k_A + 1$.

(c). All actions of the form x_k with $k_D \leq k \leq \frac{n}{2} - a$ and $\frac{n}{2} + a + 1 \leq k \leq n - k_D + 1$ are best responses for D to A's strategy.

(d). No actions of the form x_k with $k < k_D$ or $k > n - k_D + 1$ are best responses for D to A's strategy.

(e). $k_D + a - 2 \leq k_A \leq k_D + a$.

Parts (a)-(d) of this theorem are immediate consequences of Theorems 1-5. These results indicate that there is a range of moderate policies with no gaps that are optimal for the advantaged candidate. There is also a range of liberal policies with no gaps and a corresponding range of conservative policies with no gaps that are optimal actions for the disadvantaged candidate.

The only part of this theorem that is significantly different from the previous results is part (e). This result indicates that there are predictable bounds on the range of optimal actions for both players in equilibrium. In particular, the lower bound on the range of optimal actions for A is approximately a grid points closer to the center than the lower bound on the range of optimal actions for D.

This result makes sense given the incentives faced by the players. D wishes to choose an action as close to A's action as possible without choosing an action so close that D loses with certainty. Now the only actions A takes with positive probability are of the form x_i with $i \geq k_A$. Thus no action of the form x_k with $k < k_A - a$ is ever a best response for D to A's strategy, as D could instead take the action x_{k_A} which is closer to all actions A takes with positive probability without being so close that D would every lose with certainty against any of these actions. From this it follows that $k_D \geq k_A - a$ or $k_A \leq k_D + a$. Similar reasoning gives the bound $k_D + a - 2 \leq k_A$.

Now I address how many policies the candidates must randomize amongst in the limit of an arbitrarily fine policy space for a fixed size of the advantaged candidate's advantage, δ. Since the advantaged candidate randomizes amongst the most moderate policies in the policy space, in order to determine how many policies the advantaged candidate uses, it suffices to find the most extreme policies that the advantaged candidate chooses in equilibrium. Such a policy is given by the following definition.

Definition. $\underline{x}(n, \delta)$ *is the most liberal policy that A must choose in equilibrium for given values of n and* δ. *Formally,* $\underline{x}(n, \delta) \equiv \min\{x_i \in X | \sigma \in \Sigma^A \Rightarrow \exists\, k \leq i \text{ for which } \sigma_k > 0\}$.

In this definition, Σ^A, the set of symmetric maxminimizer strategies for candidate A, is implicitly a function of n and δ. I now note how the most liberal policy that A must choose in equilibrium varies with δ as the number of policies in the policy space becomes large. Throughout I restrict attention to values of n for which n is even and δ is not an integral multiple of $\frac{1}{n-1}$.

Theorem 7. $\limsup_{n\to\infty} \underline{x}(n, \delta) \leq \max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}$.

Thus in the limit of an arbitrarily fine policy space, the advantaged candidate must use policies that are at least as liberal as $\max\{\frac{1+\delta}{3}, \frac{1}{2}-\delta\}$, and the advantaged candidate uses the policies in the interval $[\max\{\frac{1+\delta}{3}, \frac{1}{2}-\delta\}, \min\{\frac{2-\delta}{3}, \frac{1}{2}+\delta\}]$. The advantaged candidate thus randomizes amongst a non-vanishing fraction of the policies in the policy space as the number of policies in the policy space becomes large. Similarly, the disadvantaged candidate will also randomize amongst a non-vanishing fraction of the policies in the policy space in order to create incentives for the advantaged candidate to do the same. The candidates will also randomize amongst a non-vanishing fraction of the policies in the policy space for any symmetric and weakly single-peaked distribution of the median voter's ideal point, though the precise number of policies that the candidates use may depend on the distribution.

To understand the intuition behind why the advantaged candidate cannot randomize amongst an infinitesimal fraction of the policies in the policy space, suppose instead that the advantaged candidate only randomized amongst policies arbitrarily close to $\frac{1}{2}$. In that case, any best response for the disadvantaged candidate would be to choose policies extremely close to either $\frac{1}{2}-\delta$ or $\frac{1}{2}+\delta$, while still choosing policies that would win with positive probability against the advantaged candidate's policies.

When the disadvantaged candidate is using such a narrow range of policies, the advantaged candidate can exploit this by choosing policies that would win with certainty against the disadvantaged candidate's policy choices. In particular, the advantaged candidate can choose a policy relatively closer to $\frac{1}{2}-\delta$ that would win with probability 1 against all liberal actions the disadvantaged candidate chooses with positive probability, while still only winning slightly less often against the conservative actions the disadvantaged candidate chooses with positive probability.

Since this change would afford the advantaged candidate a higher payoff than choosing policies arbitrarily close to $\frac{1}{2}$, there is no equilibrium in which the advantaged candidate only chooses policies arbitrarily close to $\frac{1}{2}$. Similar reasoning explains why the advantaged candidate uses policies as liberal as $\max\{\frac{1+\delta}{3}, \frac{1}{2}-\delta\}$ in the limit of an arbitrarily large policy space.

Finally, I give a useful lower bound on the probability the advantaged candidate wins the election.

Theorem 8. *The probability the advantaged candidate wins the election is at least* $\frac{1}{2}+\frac{a}{n}$.

This result follows by noting that the advantaged candidate can guarantee that he or she will win with probability $\frac{1}{2} + \frac{a}{n}$ by using the strategy $\sigma^{n/2}$. Because of this, any maxminimizer strategy for the advantaged candidate will win the election with at least probability $\frac{1}{2} + \frac{a}{n}$.

This bound will also hold for any symmetric and weakly single-peaked distribution of the median voter's ideal point. In fact, for these more general distributions of the median voter's ideal point, one can typically prove a stronger lower bound on the probability with which the advantaged candidate wins the election. For these more general distributions, there will typically be a higher probability that the median voter's ideal point is near the center of the policy space. Since the advantaged candidate wins when the median voter's ideal point is near the center of the policy space, the advantaged candidate will typically win with greater probability for these more general distributions of the median voter's ideal point.

5. Conclusion

This paper has characterized the optimal actions for candidates in a particular mixed strategy equilibrium that may arise in a Downsian model in which one candidate is endowed with some valence advantage that is so large that voters might prefer this candidate even if they prefer the other candidate's policy positions. In this equilibrium, there is a range of moderate policies with no gaps that are optimal actions for the advantaged candidate, but all other policies are not optimal for this candidate. There is also a range of liberal policies with no gaps and a corresponding range of conservative policies with no gaps that are optimal actions for the disadvantaged candidate. The most moderate policy guaranteed to be an optimal action for the disadvantaged candidate, $x_{n/2-a}$, becomes less moderate as the size of the advantaged candidate's advantage increases. And the most extreme policy that is optimal for the advantaged candidate, x_{k_A}, is more moderate than the most extreme policy that is optimal for the disadvantaged candidate, x_{k_D}, by an amount approximately proportional to the size of the advantaged candidate's advantage.

It is worth noting that the results when a candidate has a large valence advantage can differ significantly from the results in Aragones and Palfrey (2002) when a candidate has a minimal valence advantage. When the size of a candidate's valence advantage is minimal, both candidates randomize amongst virtually identical sets of policies. By contrast, if the advantaged candidate has a large valence advantage, then there need not be any overlap between the optimal actions for the advantaged candidate and the optimal actions for the disadvantaged candidate. Furthermore, in Aragones and Palfrey (2002), when the number of points in the policy space becomes large, both candidates only use policies arbitrarily close to $\frac{1}{2}$, and the advantaged candidate wins with probability arbitrarily close to $\frac{1}{2}$. By contrast, Theorems 7 and 8 in the present paper indicate that when a candidate's valence advantage δ is bounded away from zero (and thus $\frac{a}{n}$ is bounded away from zero for all n), then the advantaged candidate uses policies bounded away from $\frac{1}{2}$, and the advantaged candidate wins with probability bounded away from $\frac{1}{2}$.

While this paper has characterized the optimal actions for the candidates in equilibrium, this paper has not calculated the precise mixed strategies that are used in this equilibrium. However, characterizing the optimal actions is an important part of a full equilibrium characterization, as this gives useful information about what actions the candidates will choose with positive probability in equilibrium. Furthermore, knowing that the optimal actions are of the form in Theorem 6 gives a useful way to calculate the mixing probabilities that the candidates would use in equilibrium. For example, to find an equilibrium strategy for the advantaged candidate, one only need find mixing probabilities such that the disadvantaged candidate is indifferent between adjacent policies in a range of policies of the form in Theorem 6(c). As Aragones and Palfrey (2002) illustrate how to ensure that candidates are indifferent between adjacent policies, one can use similar techniques to derive the advantaged candidate's mixing probabilities here.

The results in my paper indicate that the disadvantaged candidate will randomize between choosing a range of policies that are relatively liberal and a range of policies that are relatively conservative. This result might seem like a poor theoretical prediction since one would not expect many candidates to randomize between choosing a liberal policy and a conservative policy. However, there are very natural empirical referents for this result. One standard interpretation of a mixed strategy is that the other players face uncertainty about what actions the player using the mixed strategy will take and the mixing probabilities reflect the probabilities with which the other players believe this player will take the various actions (Harsanyi, 1973; Rubinstein, 1991).

This interpretation is well-suited to one of the most natural applications of the model. One of the most common situations in which a candidate has a valence advantage is when an incumbent politician runs against challenger and the incumbent has an incumbency advantage. In this case, if the challenger has not previously run for public office and not publicly stated his or her general policy leanings, the incumbent may not be at all sure as to whether the challenger will choose a liberal policy or a conservative policy. The incumbent may then very well believe there is a chance the challenger will choose a liberal policy and a chance the challenger will choose a conservative policy. Thus the mixed strategy for the disadvantaged candidate corresponds very naturally to beliefs an incumbent may have about a challenger's likely policy selection.

Another natural application of the results is for party primaries. In a Democratic party primary, a candidate running for office for the first time might not particularly care whether he or she runs as a liberal Democrat or a conservative Democrat as long as he or she runs as a Democrat. However, the candidate may feel a need to choose a different policy than an incumbent that is running as a moderate Democrat. The mixed strategies in this paper for the disadvantaged candidate may then correspond naturally to the mixed strategies a disadvantaged candidate would use in a party primary.

It is also worth noting that the equilibrium characterization given here predicts that the optimal actions for the advantaged candidate are more moderate than the optimal actions for the disadvantaged candidate. Moreover, the most moderate policy that is guaranteed to be an optimal action for the disadvantaged candidate, $x_{n/2-a}$, is relatively further from the center of the policy space for larger sizes of the advantaged candidate's advantage. These results correspond well to empirical evidence. For example, Ansolabehere *et al.* (2001), Fiorina (1973), and Stone and Simas (2007) conduct empirical studies on how the size of a candidate's incumbency advantage affects whether candidates choose relatively moderate or extreme policies. These authors note that candidates tend to assume more moderate policies as the size of their valence advantages increase and tend to choose more extreme policies when they face relatively larger disadvantages. The predictions of the model are thus supported by these empirical studies.

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Appendix

Theorem 1. *There is an equilibrium in which both candidates use symmetric strategies.*

Proof. Since this game is finite strategic game, we know from Proposition 33.1 of Osborne and Rubinstein (1994) that there exists an equilibrium in mixed strategies. Let (σ^A, σ^D) be some such equilibrium, and consider the strategy $\sigma^{A'}$ defined by setting $\sigma_i^{A'} = \sigma_{n-i+1}^A$ for all i. By the symmetry of the players' payoff functions, we know that $\Pi_A(\sigma^A, x_k) = \Pi_A(\sigma^{A'}, x_{n-k+1})$ for all k. Thus since $\Pi_A(\sigma^A, x_k) \geq \overline{\Pi}_A$ for all k, we have $\Pi_A(\sigma^{A'}, x_k) \geq \overline{\Pi}_A$ for all k as well. So the fact that σ^A is a maxminimizer strategy implies $\sigma^{A'}$ is a maxminimizer strategy.

Now consider the strategy $\sigma^{A''}$ defined by $\sigma^{A''} = \frac{1}{2}(\sigma^A + \sigma^{A'})$. Since the set of maxminimizer strategies for A is convex, $\sigma^{A''}$ is also a maxminimizer strategy for A. And $\sigma^{A''}$ is a symmetric strategy because $\sigma_i^{A''} = \frac{1}{2}(\sigma_i^A + \sigma_i^{A'}) = \frac{1}{2}(\sigma_{n-i+1}^{A'} + \sigma_{n-i+1}^A) = \sigma_{n-i+1}^{A''}$. From this it follows that $\sigma^{A''}$ is a symmetric maxminimizer strategy.

A virtually identical argument shows that if $\sigma^{D'}$ is defined by setting $\sigma_i^{D'} = \sigma_{n-i+1}^D$ for all i and $\sigma^{D''} = \frac{1}{2}(\sigma^D + \sigma^{D'})$, then $\sigma^{D''}$ is a symmetric maxminimizer strategy for D. But this means that $(\sigma^{A''}, \sigma^{D''})$ is an equilibrium in which both candidates use symmetric strategies. The result follows. □

Lemma 1. *Suppose D chooses an action x_k with $k \leq \frac{n}{2}$. Then we have the following:*

(a). All strategies of the form σ^i for any integer i satisfying $k + a \leq i \leq \frac{n}{2}$ afford the same expected payoff for A against D's action.

(b). For any positive integer $i \leq k - a + 1$, A's expected payoff from using the strategy σ^i is strictly increasing in i.

(c). For any positive integer i satisfying $i \leq k + a - 1$ and $i \leq \frac{n}{2}$, A's expected payoff from using the strategy σ^i is nondecreasing in i.

Proof. (a). To prove this it suffices to show that $\Pi_A(\sigma^i, x_k) = \Pi_A(\sigma^{i+1}, x_k)$ for all integers i satisfying $k + a \leq i < \frac{n}{2}$. Note that $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^i, x_k) = \Pi_A(\sigma^{i+1}, x_k)$ holds if and only if $\frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k)) = \frac{1}{2}(\pi_A(x_{i+1}, x_k) + \pi_A(x_{n-i}, x_k))$ or $\pi_A(x_i, x_k) - \pi_A(x_{i+1}, x_k) = \pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k)$. It thus suffices to show that $\pi_A(x_i, x_k) - \pi_A(x_{i+1}, x_k) = \pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k)$ for all integers i satisfying $k + a \leq i < \frac{n}{2}$. I consider two cases:

Case 1: Suppose $k = i - a - 2j + 1$ for some positive integer j. In this case, we know from equation (1) that $\pi_A(x_i, x_k) = \frac{n-i+a+j}{n}$ and $\pi_A(x_{i+1}, x_k) = \frac{n-(i+1)+a+j}{n}$ for this integer j. Thus $\pi_A(x_i, x_k) - \pi_A(x_{i+1}, x_k) = \frac{1}{n}$. Also, $k \equiv n - i - a + 1 \pmod 2$ and $k = n - i - a - 2j' + 1$ for some positive integer j'. But if this holds, then we know from equation (1) that $\pi_A(x_{n-i}, x_k) = \frac{n-(n-i)+a+j'}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j'}{n}$ for this integer j'. Thus $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) = \frac{1}{n}$ as well and $\pi_A(x_i, x_k) - \pi_A(x_{i+1}, x_k) = \pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k)$.

Case 2: Suppose $k = i - a - 2j + 1$ does not hold for any positive integer j. In this case, we have $k = i-a-2j$ for some nonnegative integer j and $k = (i+1)-a-2(j+1)+1$ for this same integer j. We then know from equation (1) that $\pi_A(x_i, x_k) = \frac{n-i+a+j}{n}$ and $\pi_A(x_{i+1}, x_k) = \frac{n-(i+1)+a+(j+1)}{n} = \frac{n-i+a+j}{n}$ for this integer j. Thus $\pi_A(x_i, x_k) - \pi_A(x_{i+1}, x_k) = 0$.

Since $k = i - a - 2j$ for some nonnegative integer j, $k \equiv n - i - a \pmod 2$, $k = n - i - a - 2j'$ for some nonnegative integer j', and $k = n - i + 1 - a - 2(j' + 1) + 1$ for this same integer j'. We then know from equation (1) that $\pi_A(x_{n-i}, x_k) = \frac{n-(n-i)+a+j'}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+(j'+1)}{n} = \frac{n-(n-i)+a+j'}{n}$ for this integer j'. Thus $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) = 0$ as well. In either case, we have $\pi_A(x_i, x_k) - \pi_A(x_{i+1}, x_k) = \pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k)$, and the result holds.

(b). To prove this result it suffices to show that $\Pi_A(\sigma^{i+1}, x_k) - \Pi_A(\sigma^i, x_k) > 0$ for all positive integers i satisfying $1 \leq i < k - a + 1$. Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^{i+1}, x_k) - \Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_{i+1}, x_k) + \pi_A(x_{n-i}, x_k)) - \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k)) = \frac{1}{2}(\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k)) + \frac{1}{2}(\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k))$. To prove the result it thus suffices to show that either $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) > 0$ and $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) = 0$ or $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) = 0$ and $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) > 0$ for all positive integers i satisfying $1 \leq i < k - a + 1$. I consider two cases:

Case 1: Suppose $k = i + a + 2j - 1$ for some positive integer j. Then $k = i + 1 + a + 2(j - 1)$ for the same integer j and we know from equation (1) that $\pi_A(x_i, x_k) = \frac{i+a+j-1}{n}$ and $\pi_A(x_{i+1}, x_k) = \frac{i+1+a+(j-1)-1}{n}$ for this integer j. Thus $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) = 0$. Also, $k \equiv n - i + 1 - a$ (mod 2), and $k = n - i + 1 - a - 2j'$ for some positive integer j'. But if this holds, then we know from equation (1) that $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j'}{n}$ and $\pi_A(x_{n-i}, x_k) = \frac{n-(n-i)+a+j'}{n}$ for this integer j'. Thus $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_{k+1}) = \frac{1}{n}$. From this it follows that $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) = 0$ and $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) > 0$ holds in this case.

Case 2: Suppose $k = i + a + 2j - 1$ does not hold for any positive integer j. In this case, we have $k = i + a + 2j$ for some nonnegative integer j and $k = i + 1 + a + 2j - 1$ for this same integer j. Thus we know from equation (1) that $\pi_A(x_i, x_k) = \frac{i+a+j-1}{n}$. Also, if $j > 0$, then

$\pi_A(x_{i+1}, x_k) = \frac{i+1+a+j-1}{n}$ for this integer j, and if $j = 0$, then $\pi_A(x_{i+1}, x_k) = 1$. In either case, it follows that $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) > 0$.

Since $k = i + a + 2j$ for some nonnegative integer j, $k \equiv n - i - a \pmod 2$, $k = n - i - a - 2j'$ for some nonnegative integer j', and $k = n - i + 1 - a - 2(j' + 1) + 1$ for this same integer j'. Thus we know from equation (1) that $\pi_A(x_{n-i}, x_k) = \frac{n-(n-i)+a+j'}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+(j'+1)}{n}$ for this integer j', meaning $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) = 0$. Combining this with the results from Case 1 shows that either $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) > 0$ and $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) = 0$ or $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) = 0$ and $\pi_A(x_{n-i}, x_k) - \pi_A(x_{n-i+1}, x_k) > 0$ for all positive integers i satisfying $1 \leq i < k - a + 1$. The result follows.

(c). We know from part (b) that A's expected payoff from using the strategy σ^i is nondecreasing in i for all positive integers $i \leq k - a + 1$. Thus to prove the result it suffices to show that if i is a positive integer satisfying $k - a + 1 \leq i < k + a - 1$ and $i < \frac{n}{2}$, then $\Pi_A(\sigma^{i+1}, x_k) \geq \Pi_A(\sigma^i, x_k)$.

Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^{i+1}, x_k) \geq \Pi_A(\sigma^i, x_k)$ holds if and only if $\frac{1}{2}(\pi_A(x_{i+1}, x_k) + \pi_A(x_{n-i}, x_k)) \geq \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$ or $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) \geq \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i}, x_k)$. It thus suffices to show that $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) \geq 0 \geq \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i}, x_k)$ for all positive integers i satisfying $k - a + 1 \leq i < k + a - 1$ and $i < \frac{n}{2}$.

Now if $k - a + 1 \leq i < k + a - 1$, then $k - a + 1 \leq i + 1 \leq k + a - 1$, $i + 1 - a + 1 \leq k \leq i + 1 + a - 1$, and we know from equation (1) that $\pi_A(x_{i+1}, x_k) = 1$. Thus $\pi_A(x_{i+1}, x_k) \geq \pi_A(x_i, x_k)$ and $\pi_A(x_{i+1}, x_k) - \pi_A(x_i, x_k) \geq 0$. It thus suffices to show that $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i}, x_k) \leq 0$ if $i < \frac{n}{2}$. I consider several cases:

<u>Case 1</u>: Suppose $k \geq n - i - a + 1$. In this case, we know from equation (1) that $\pi_A(x_{n-i}, x_k) = 1$ and $\pi_A(x_{n-i+1}, x_k) \leq \pi_A(x_{n-i}, x_k)$. Thus $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i}, x_k) \leq 0$.

<u>Case 2</u>: Suppose $k = n - i - a - 2j + 1$ for some positive integer j. Then we know from equation (1) that $\pi_A(x_{n-i}, x_k) = \frac{n-(n-i)+a+j}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j}{n}$ for this integer j. Thus $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i}, x_k) = -\frac{1}{n} \leq 0$.

<u>Case 3</u>: Suppose neither of the first two cases hold. Then $k = n - i - a - 2j$ for some nonnegative integer j, and $k = n - i + 1 - a - 2(j + 1) + 1$ for this same integer j. Then we know from equation (1) that $\pi_A(x_{n-i}, x_k) = \frac{n-(n-i)+a+j}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j+1}{n} = \frac{n-(n-i)+a+j}{n}$ for this integer j. Thus $\pi_A(x_{n-i+1}, x_k) = \pi_A(x_{n-i}, x_k)$, and we always have $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i}, x_k) \leq 0$. The result then follows.

$\square$

Lemma 2. *Suppose A uses the strategy σ^i for some $i \leq \frac{n}{2}$. Then we have the following:*

(a). For any positive integer $k \leq i-a$, D's expected payoff from choosing the action x_k is strictly increasing in k.

(b). All actions of the form x_k for positive integers k satisfying $i+a \leq k \leq n-i+1-a$ afford D the same expected payoff against A's strategy.

(c). For any positive integer k satisfying $i-a+1 \leq k \leq n-i+1-a$, D's expected payoff from choosing the action x_k is nondecreasing in k.

(d). For any positive integer k satisfying $i-a+1 \leq k \leq i+a-1$ and $k+2 \leq n-i+1-a$, D's expected payoff from choosing the action x_{k+2} is strictly greater than D's expected payoff from choosing the action x_k.

Proof. (a). To prove this it suffices to show that for all integers k satisfying $1 < k \leq i-a$, we have $\Pi_D(\sigma^i, x_k) - \Pi_D(\sigma^i, x_{k-1}) = \Pi_A(\sigma^i, x_{k-1}) - \Pi_A(\sigma^i, x_k) > 0$. Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^i, x_{k-1}) - \Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_{k-1}) + \pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_i, x_k) - \pi_A(x_{n-i+1}, x_k)) = \frac{1}{2}[(\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k)) + (\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k))]$. It thus suffices to show that either $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) > 0$ and $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) = 0$ or $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) = 0$ and $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) > 0$ for all integers k satisfying $1 < k \leq i-a$. I consider two cases:

Case 1: Suppose $k = i-a-2j+1$ for some positive integer j. Then $k-1 = i-a-2j$ for the same integer j and we know from equation (1) that $\pi_A(x_i, x_k) = \pi_A(x_i, x_{k-1}) = \frac{n-i+a+j}{n}$ for this integer j. Thus $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) = 0$. Also, $k \equiv n-i+1-a \pmod 2$, $k = n-i+1-a-2j'$ for some positive integer j', and $k-1 = n-i+1-a-2(j'+1)+1$ for the same positive integer j'. But if this holds, then we know from equation (1) that $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j'}{n}$ and $\pi_A(x_{n-i+1}, x_{k-1}) = \frac{n-(n-i+1)+a+(j'+1)}{n}$ for this integer j'. Thus $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) = \frac{1}{n}$. So in this case $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) = 0$ and $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) > 0$.

Case 2: Suppose $k = i-a-2j+1$ does not hold for any positive integer j. In this case, we have $k = i-a-2j$ for some nonnegative integer j and $k-1 = i-a-2(j+1)+1$ for this same integer j. Thus we know from equation (1) that $\pi_A(x_i, x_k) = \frac{n-i+a+j}{n}$ and $\pi_A(x_i, x_{k-1}) = \frac{n-i+a+(j+1)}{n}$ for this integer j. From this it follows that $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) = \frac{1}{n}$.

Since $k = i-a-2j$ for some nonnegative integer j, $k \equiv n-i-a \pmod 2$, $k = n-i+1-a-2j'+1$ for some nonnegative integer j', and $k-1 = n-i+1-a-2j'$ for this same integer j'. Thus we know from equation (1) that $\pi_A(x_{n-i+1}, x_k) = \pi_A(x_{n-i+1}, x_{k-1}) = \frac{n-(n-i+1)+a+j'}{n}$ for this integer j' and $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) = 0$. Combining this with the results from Case 1 shows that either $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) = 0$ and $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) > 0$ or $\pi_A(x_i, x_{k-1}) - \pi_A(x_i, x_k) > 0$ and $\pi_A(x_{n-i+1}, x_{k-1}) - \pi_A(x_{n-i+1}, x_k) = 0$. The result then follows.

(b). To prove this it suffices to show that if k is a positive integer satisfying $i + a \leq k$ and $k+1 \leq n-i+1-a$, then $\Pi_A(\sigma^i, x_k) = \Pi_A(\sigma^i, x_{k+1})$. Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^i, x_k) = \Pi_A(\sigma^i, x_{k+1})$ holds if and only if $\frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k)) = \frac{1}{2}(\pi_A(x_i, x_{k+1}) + \pi_A(x_{n-i+1}, x_{k+1}))$ or $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) = \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1})$. It thus suffices to show that $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) = \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1})$ for all positive integers k satisfying $i + a \leq k$ and $k + 1 \leq n - i + 1 - a$. I consider two cases:

<u>Case 1</u>: Suppose $k = i + a + 2j - 1$ for some positive integer j. Then $k + 1 = i + a + 2j$ for the same integer j and we know from equation (1) that $\pi_A(x_i, x_{k+1}) = \pi_A(x_i, x_k) = \frac{i+a+j-1}{n}$ for this integer j. Thus $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) = 0$. Also, $k \equiv n-i+1-a \pmod 2$, $k = n-i+1-a-2j'$ for some positive integer j', and $k+1 = n-i+1-a-2j'+1$ for the same positive integer j'. But if this holds, then we know from equation (1) that $\pi_A(x_{n-i+1}, x_k) = \pi_A(x_{n-i+1}, x_{k+1}) = \frac{n-(n-i+1)+a+j'}{n}$ for this integer j'. Thus $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1}) = 0$ and $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) = \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1})$ holds in this case.

<u>Case 2</u>: Suppose $k = i+a+2j-1$ does not hold for any positive integer j. In this case, we have $k = i+a+2j$ for some nonnegative integer j and $k+1 = i+a+2(j+1)-1$ for this same integer j. Thus we know from equation (1) that $\pi_A(x_i, x_k) = \frac{i+a+j-1}{n}$ and $\pi_A(x_i, x_{k+1}) = \frac{i+a+(j+1)-1}{n}$ for this integer j. From this it follows that $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) = \frac{1}{n}$.

Since $k = i+a+2j$ for some nonnegative integer j, $k \equiv n-i-a \pmod 2$, $k = n-i+1-a-2j'+1$ for some positive integer j', and $k+1 = n-i+1-a-2(j'-1)$ for this same integer j'. Thus we know from equation (1) that $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j'}{n}$ and $\pi_A(x_{n-i+1}, x_{k+1}) = \frac{n-(n-i+1)+a+(j'-1)}{n}$ for this integer j'. From this it follows that $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1}) = \frac{1}{n} = \pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k)$. Combining this with the results from Case 1 shows that $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) = \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1})$ for all positive integers k satisfying $i + a \leq k$ and $k + 1 \leq n - i + 1 - a$. The result then follows.

(c). First note from part (b) that D's expected payoff from choosing the action x_k is constant in k for any k in the range $i+a \leq k \leq n-i+1-a$. Thus the result holds for all positive integers k in the range $i+a \leq k \leq n-i+1-a$. So to prove the result, it suffices to show that if k is a positive integer satisfying $i-a+1 \leq k \leq i+a-1$ and $k+1 \leq n-i+1-a$, then $\Pi_D(\sigma^i, x_{k+1}) \geq \Pi_D(\sigma^i, x_k)$ or $\Pi_A(\sigma^i, x_k) \geq \Pi_A(\sigma^i, x_{k+1})$.

Note that $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^i, x_k) \geq \Pi_A(\sigma^i, x_{k+1})$ holds if and only if $\frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k)) \geq \frac{1}{2}(\pi_A(x_i, x_{k+1}) + \pi_A(x_{n-i+1}, x_{k+1}))$ or $\pi_A(x_i, x_k) - \pi_A(x_i, x_{k+1}) \geq \pi_A(x_{n-i+1}, x_{k+1}) - \pi_A(x_{n-i+1}, x_k)$. It thus suffices to show that $\pi_A(x_i, x_k) - \pi_A(x_i, x_{k+1}) \geq 0 \geq \pi_A(x_{n-i+1}, x_{k+1}) - \pi_A(x_{n-i+1}, x_k)$ for all integers k satisfying $i-a+1 \leq k \leq i+a-1$ and $k+1 \leq n-i+1-a$.

Note that if k satisfies $i-a+1 \leq k \leq i+a-1$, then we know from equation (1) that $\pi_A(x_i, x_k) = 1$. From this it follows that if k satisfies $i-a+1 \leq k \leq i+a-1$, then $\pi_A(x_i, x_k) \geq \pi_A(x_i, x_{k+1})$ and $\pi_A(x_i, x_k) - \pi_A(x_i, x_{k+1}) \geq 0$.

Now I show that $\pi_A(x_{n-i+1}, x_{k+1}) - \pi_A(x_{n-i+1}, x_k) \leq 0$ if $k+1 \leq n-i+1-a$. I consider two cases:

<u>Case 1</u>: Suppose $k+1 = n-i+1-a-2j+1$ for some positive integer j. In this case, we have $k = n-i+1-a-2j$ for this same integer j and we know from equation (1) that $\pi_A(x_{n-i+1}, x_{k+1}) = \pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j}{n}$ for this integer j. Thus $\pi_A(x_{n-i+1}, x_{k+1}) - \pi_A(x_{n-i+1}, x_k) \leq 0$ holds in this case.

<u>Case 2</u>: Suppose $k+1 = n-i+1-a-2j+1$ does not hold for any positive integer j. In this case, we have $k+1 = n-i+1-a-2j$ for some nonnegative integer j and $k = n-i+1-a-2(j+1)+1$ for this same integer j. Thus we know from equation (1) that $\pi_A(x_{n-i+1}, x_{k+1}) = \frac{n-(n-i+1)+a+j}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+(j+1)}{n}$ for this integer j and $\pi_A(x_{n-i+1}, x_{k+1}) - \pi_A(x_{n-i+1}, x_k) = -\frac{1}{n} \leq 0$. Thus $\pi_A(x_{n-i+1}, x_{k+1}) - \pi_A(x_{n-i+1}, x_k) \leq 0$ holds if $k+1 \leq n-i+1-a$. The result follows.

(d). To prove the result, it suffices to show that if k is a positive integer satisfying $i-a+1 \leq k \leq i+a-1$ and $k+2 \leq n-i+1-a$, then $\Pi_D(\sigma^i, x_{k+2}) > \Pi_D(\sigma^i, x_k)$ or $\Pi_A(\sigma^i, x_k) > \Pi_A(\sigma^i, x_{k+2})$. Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^i, x_k) > \Pi_A(\sigma^i, x_{k+2})$ holds if and only if $\frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k)) > \frac{1}{2}(\pi_A(x_i, x_{k+2}) + \pi_A(x_{n-i+1}, x_{k+2}))$ or $\pi_A(x_i, x_k) - \pi_A(x_i, x_{k+2}) > \pi_A(x_{n-i+1}, x_{k+2}) - \pi_A(x_{n-i+1}, x_k)$. It thus suffices to show that $\pi_A(x_i, x_k) - \pi_A(x_i, x_{k+2}) \geq 0 >$

$\pi_A(x_{n-i+1}, x_{k+2}) - \pi_A(x_{n-i+1}, x_k)$ for all integers k satisfying $i - a + 1 \leq k \leq i + a - 1$ and $k + 2 \leq n - i + 1 - a$.

Note that if k satisfies $i-a+1 \leq k \leq i+a-1$, then we know from equation (1) that $\pi_A(x_i, x_k) = 1$. From this it follows that if k satisfies $i - a + 1 \leq k \leq i + a - 1$, then $\pi_A(x_i, x_k) \geq \pi_A(x_i, x_{k+2})$ and $\pi_A(x_i, x_k) - \pi_A(x_i, x_{k+2}) \geq 0$.

Now I show that $\pi_A(x_{n-i+1}, x_{k+2}) - \pi_A(x_{n-i+1}, x_k) < 0$ if $k + 2 \leq n - i + 1 - a$. I consider two cases:

Case 1: Suppose $k + 2 = n - i + 1 - a - 2j + 1$ for some positive integer j. In this case, we have $k = n - i + 1 - a - 2(j + 1) + 1$ for this same integer j and we know from equation (1) that $\pi_A(x_{n-i+1}, x_{k+2}) = \frac{n-(n-i+1)+a+j}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j+1}{n}$ for this integer j. Thus $\pi_A(x_{n-i+1}, x_{k+2}) - \pi_A(x_{n-i+1}, x_k) = -\frac{1}{n} < 0$ holds in this case.

Case 2: Suppose $k + 2 = n - i + 1 - a - 2j + 1$ does not hold for any positive integer j. In this case, we have $k+2 = n-i+1-a-2j$ for some nonnegative integer j and $k = n-i+1-a-2(j+1)$ for this same integer j. Thus we know from equation (1) that $\pi_A(x_{n-i+1}, x_{k+2}) = \frac{n-(n-i+1)+a+j}{n}$ and $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+(j+1)}{n}$ for this integer j, meaning $\pi_A(x_{n-i+1}, x_{k+2}) - \pi_A(x_{n-i+1}, x_k) = -\frac{1}{n} < 0$. In either case, we have $\pi_A(x_{n-i+1}, x_{k+2}) - \pi_A(x_{n-i+1}, x_k) < 0$ and the result holds.

□

Lemma 3. *There is an equilibrium in which A uses a symmetric strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that $\sigma_i = 0$ for all positive integers i satisfying $1 \leq i \leq a - 1$.*

Proof. Consider some $\sigma \in \Sigma^A$. Since σ is a symmetric strategy, we can write $\sigma = \sum_{i=1}^{n/2} 2\sigma_i \sigma^i$. Now consider the alternative strategy σ' given by $\sigma' = \sum_{i=1}^{a-1} 2\sigma_i \sigma^a + \sum_{i=a}^{n/2} 2\sigma_i \sigma^i$. Note that σ' is a symmetric strategy with $\sigma_i = 0$ for all positive integers i satisfying $1 \leq i \leq a - 1$. Thus to prove the result it suffices to show that σ' is also a maxminimizer strategy. And to prove this it suffices to show that $\Pi_A(\sigma', x_k) \geq \Pi_A(\sigma, x_k)$ for all k. I thus seek to prove $\Pi_A(\sigma', x_k) - \Pi_A(\sigma, x_k) \geq 0$ for all k.

Note that $\sigma' - \sigma = \sum_{i=1}^{a-1} 2\sigma_i(\sigma^a - \sigma^i)$. Thus $\Pi_A(\sigma', x_k) - \Pi_A(\sigma, x_k) = \sum_{i=1}^{a-1} 2\sigma_i(\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k))$, and to prove that $\Pi_A(\sigma', x_k) - \Pi_A(\sigma, x_k) \geq 0$ for all k, it suffices to show that $\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k) \geq 0$ for all k and all positive integers i satisfying $1 \leq i \leq a - 1$.

Now we know from Lemma 1(c) that if D chooses an action x_k with $k \leq \frac{n}{2}$, then A's payoff from using a strategy σ^i with $i < k + a$ and $i \leq \frac{n}{2}$ is nondecreasing in i. But for any positive integer k, we have $i < k + a$ and $i \leq \frac{n}{2}$ for all positive integers $i \leq a$. Thus for any $k \leq \frac{n}{2}$ and any positive integer i satisfying $1 \leq i \leq a-1$, we have $\Pi_A(\sigma^a, x_k) \geq \Pi_A(\sigma^i, x_k)$ and $\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k) \geq 0$.

But for any symmetric strategy σ, we have $\Pi_A(\sigma, x_k) = \Pi_A(\sigma, x_{n-k+1})$. Thus if $\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k) \geq 0$ for some k, then $\Pi_A(\sigma^a, x_{n-k+1}) - \Pi_A(\sigma^i, x_{n-k+1}) \geq 0$. But this means that if $\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k) \geq 0$ holds for all $k \leq \frac{n}{2}$ and any positive integer i satisfying $1 \leq i \leq a-1$, then $\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k) \geq 0$ also holds for all $k > \frac{n}{2}$ and any positive integer i satisfying $1 \leq i \leq a-1$. But then $\Pi_A(\sigma^a, x_k) - \Pi_A(\sigma^i, x_k) \geq 0$ holds for all k and all positive integers i satisfying $1 \leq i \leq a-1$. The result then follows.

□

Lemma 4. *Suppose A uses a symmetric strategy σ that has a gap at x_i for some positive integer i satisfying $a+1 \leq i \leq \frac{n}{2}$. Also suppose that $\sigma_{i-1} > 0$ for this i. Then it is not a best response for D to take the action x_{i-a}.*

Proof. To prove this it suffices to show that either $\Pi_D(\sigma, x_{i-a+1}) > \Pi_D(\sigma, x_{i-a})$ or $\Pi_D(\sigma, x_{i-a+2}) > \Pi_D(\sigma, x_{i-a})$ if A uses a symmetric strategy σ that has a gap at x_i for some positive integer i satisfying $a+1 \leq i \leq \frac{n}{2}$. Now since σ is a symmetric strategy, we can write $\sigma = \sum_{j=1}^{n/2} 2\sigma_j \sigma^j$. And since σ has a gap at x_i, we know that $\sigma_i = 0$. Thus $\sigma = \sum_{j=1}^{i-1} 2\sigma_j \sigma^j + \sum_{j=i+1}^{n/2} 2\sigma_j \sigma^j$ and $\Pi_D(\sigma, x_k) = \sum_{j=1}^{i-1} 2\sigma_j \Pi_D(\sigma^j, x_k) + \sum_{j=i+1}^{n/2} 2\sigma_j \Pi_D(\sigma^j, x_k)$ for all k. I consider two cases:

<u>Case 1</u>: Suppose $\sigma_j > 0$ for some positive integer j satisfying $i+1 \leq j \leq \frac{n}{2}$. We know from Lemma 2(a) that $\Pi_D(\sigma^j, x_{i-a+1}) > \Pi_D(\sigma^j, x_{i-a})$ for any such j. And we also know from Lemma 2(c) that $\Pi_D(\sigma^j, x_{i-a+1}) \geq \Pi_D(\sigma^j, x_{i-a})$ for any $j \leq i-1$. Combining these results with the fact that $\Pi_D(\sigma, x_k) = \sum_{j=1}^{i-1} 2\sigma_j \Pi_D(\sigma^j, x_k) + \sum_{j=i+1}^{n/2} 2\sigma_j \Pi_D(\sigma^j, x_k)$ for any k shows that $\Pi_D(\sigma, x_{i-a+1}) > \Pi_D(\sigma, x_{i-a})$.

<u>Case 2</u>: Suppose $\sigma_j > 0$ does not hold for any positive integer j satisfying $i+1 \leq j \leq \frac{n}{2}$. In that case, we have $\sigma_{i-1} > 0$ and $\Pi_D(\sigma, x_k) = \sum_{j=1}^{i-1} 2\sigma_j \Pi_D(\sigma^j, x_k)$. Now we know from Lemma 2(d) that $\Pi_D(\sigma^{i-1}, x_{i-a+2}) > \Pi_D(\sigma^{i-1}, x_{i-a})$. And we know from Lemma 2(c) that $\Pi_D(\sigma^j, x_{i-a+2}) \geq \Pi_D(\sigma^j, x_{i-a})$ for all $j < i-1$. Combining this with the facts that $\sigma_{i-1} > 0$ and $\Pi_D(\sigma, x_k) = \sum_{j=1}^{i-1} 2\sigma_j \Pi_D(\sigma^j, x_k)$ shows that $\Pi_D(\sigma, x_{i-a+2}) > \Pi_D(\sigma, x_{i-a})$. Thus either $\Pi_D(\sigma, x_{i-a+1}) > \Pi_D(\sigma, x_{i-a})$ or $\Pi_D(\sigma, x_{i-a+2}) > \Pi_D(\sigma, x_{i-a})$ and the result holds.

□

Theorem 2. *There is a strategy in Σ_a^A with no gaps. Furthermore, if there is a strategy $\sigma \in \Sigma_a^A$ such that $\sigma_j > 0$ and $\sigma_i = 0$ for all $i < j$ for some $j \leq \frac{n}{2}$, then for this j there is also a strategy $\sigma' \in \Sigma_a^A$ with no gaps such that $\sigma'_j > 0$ and $\sigma'_i = 0$ for all $i < j$.*

Proof. We know from Lemma 3 that there is a strategy in Σ_a^A. Any such strategy σ is a symmetric strategy so $\sigma_i > 0$ for some $i \leq \frac{n}{2}$ for any such strategy. Thus for any such strategy σ there is some $j \leq \frac{n}{2}$ such that $\sigma_j > 0$ and $\sigma_i = 0$ for all $i < j$. Consider some such j and let Σ_{aj}^A denote the set of strategies in Σ_a^A for which $\sigma_j > 0$ and $\sigma_i = 0$ for all $i < j$. To prove the result it suffices to show that there is a strategy in Σ_{aj}^A with no gaps.

Suppose by means of contradiction that there is no strategy in Σ_{aj}^A with no gaps. In that case, all strategies in Σ_{aj}^A have gaps. Let r denote the smallest integer such that there is a strategy in Σ_{aj}^A with r distinct gaps or r distinct integers i such that there is a gap at x_i, and let Σ_{ajr}^A denote the set of strategies in Σ_{aj}^A with exactly r gaps.

Now let i denote the smallest integer i such that there is a strategy in Σ_{ajr}^A with a gap at x_i. Note that $a+1 \leq i \leq \frac{n}{2}$: If there is a gap at x_i in some strategy $\sigma \in \Sigma_{ajr}^A$ for some $i > \frac{n}{2}$, then since σ is a symmetric strategy, there is also a gap at x_{n-i+1}, meaning there is a gap at some x_i with $i \leq \frac{n}{2}$. And since any strategy $\sigma \in \Sigma_{ajr}^A$ has $\sigma_i = 0$ for all positive integers i satisfying $1 \leq i \leq a-1$, the smallest i for which we can have $\sigma_i > 0$ is $i = a$ and the smallest i for which there can be a gap at x_i is $i = a+1$. Thus if i is the smallest integer such that there is a strategy in Σ_{ajr}^A with a gap at x_i, we have $a+1 \leq i \leq \frac{n}{2}$.

Now let σ denote a strategy in Σ_{ajr}^A with a gap at x_i. Since i is the smallest integer such that σ has a gap at x_i, it follows that $\sigma_{i-1} > 0$. And since $a+1 \leq i \leq \frac{n}{2}$, it follows from Lemma 4 that if A uses the strategy σ, then it is not a best response for D to choose the action x_{i-a}. In particular, if $\Pi \equiv \max_{1 \leq k \leq n} \Pi_D(\sigma, x_k)$ denotes the maximum payoff that D can achieve when A uses the strategy σ, then $\Pi_D(\sigma, x_{i-a}) < \Pi$.

Now let σ^ϵ be the strategy defined by $\sigma^\epsilon \equiv \sigma - \epsilon\sigma^{i-1} + \epsilon\sigma^i$. Since $\sigma_{i-1} > 0$, for sufficiently small $\epsilon > 0$, we have $\sigma_{i-1}^\epsilon > 0$ and $\sigma^\epsilon \geq 0$. Thus σ^ϵ is a feasible symmetric strategy. Furthermore, since $\sigma_i^\epsilon > 0$, σ^ϵ has fewer gaps than σ (and σ^ϵ may even have no gaps). Thus to obtain a contradiction, it suffices to show that σ^ϵ is a maxminimizer strategy for some sufficiently small $\epsilon > 0$ or that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all k if $\epsilon > 0$ is sufficiently small. I first show that this holds if $k \leq \frac{n}{2}$.

Since $\Pi_D(\sigma, x_k)$ is continuous in σ for all k, for sufficiently small $\epsilon > 0$, we have $\Pi_D(\sigma^\epsilon, x_{i-a}) < \Pi$. Thus to prove that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all $k \leq \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small, it suffices to show that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi_D(\sigma, x_k)$ for all $k \leq \frac{n}{2}$ with $k \neq i - a$ if $\epsilon > 0$ is sufficiently small. Now $\sigma^\epsilon - \sigma = \epsilon(\sigma^i - \sigma^{i-1})$. Thus for all k, $\Pi_D(\sigma^\epsilon, x_k) - \Pi_D(\sigma, x_k) = \epsilon(\Pi_D(\sigma^i, x_k) - \Pi_D(\sigma^{i-1}, x_k))$. So for $\epsilon > 0$, $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi_D(\sigma, x_k)$ holds if and only if $\Pi_D(\sigma^i, x_k) \leq \Pi_D(\sigma^{i-1}, x_k)$ or $\Pi_A(\sigma^i, x_k) \geq \Pi_A(\sigma^{i-1}, x_k)$. So to prove that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all $k \leq \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small, it suffices to show that $\Pi_A(\sigma^i, x_k) \geq \Pi_A(\sigma^{i-1}, x_k)$ for all $k \leq \frac{n}{2}$ with $k \neq i - a$.

Now if $i - a < k \leq \frac{n}{2}$, then we know from Lemma 1(c) that $\Pi_A(\sigma^i, x_k) \geq \Pi_A(\sigma^{i-1}, x_k)$. And if $k \leq i - a - 1$, then we know from Lemma 1(a) that $\Pi_A(\sigma^i, x_k) = \Pi_A(\sigma^{i-1}, x_k)$. Thus we have $\Pi_A(\sigma^i, x_k) \geq \Pi_A(\sigma^{i-1}, x_k)$ for all $k \leq \frac{n}{2}$ with $k \neq i - a$, and $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all $k \leq \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small.

But since σ^ϵ is a symmetric strategy, $\Pi_D(\sigma^\epsilon, x_k) = \Pi_D(\sigma^\epsilon, x_{n-k+1})$ for all k. Thus since $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all $k \leq \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small, we also have $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all k if $\epsilon > 0$ is sufficiently small. But I have indicated that this contradicts my assumption that there is no strategy in Σ^A_{aj} with no gaps. Thus there is a strategy in Σ^A_{aj} with no gaps and the result follows.

□

Theorem 3. *Let j be the smallest positive integer such that there is some $\sigma \in \Sigma^A_a$ with $\sigma_j > 0$. Then in every equilibrium (σ^A, σ^D), all actions of the form x_i with $j \leq i \leq n - j + 1$ must be best responses for A to D's strategy.*

Proof. If j is the smallest positive integer such that there is some $\sigma \in \Sigma^A_a$ with $\sigma_j > 0$, then the fact that all strategies in Σ^A_a are symmetric means that $j \leq \frac{n}{2}$. Also, since j is the smallest such integer, it follows that if $\sigma \in \Sigma^A_a$ has $\sigma_j > 0$, then $\sigma_i = 0$ for all $i < j$. We thus know from Theorem 2 that there is a strategy $\sigma^{A'} \in \Sigma^A_a$ with no gaps such that $\sigma^{A'}_j > 0$. Any such strategy takes all actions of the form x_i with $j \leq i \leq n - j + 1$ with positive probability.

Now suppose (σ^A, σ^D) is an equilibrium. In that case, $\sigma^{A'}$ is a maxminimizer strategy for A, σ^D is a maxminimizer strategy for D, and $(\sigma^{A'}, \sigma^D)$ is also an equilibrium. But $\sigma^{A'}$ is a strategy that takes all actions of the form x_i with $j \leq i \leq n - j + 1$ with positive probability. Thus all actions of the form x_i with $j \leq i \leq n - j + 1$ are best responses for A to D's strategy. But I have indicated that this holds for any equilibrium (σ^A, σ^D). Thus in every equilibrium (σ^A, σ^D), all actions of the form x_i with $j \leq i \leq n - j + 1$ must be best responses for A to D's strategy.

□

Theorem 4. *Suppose there is a strategy in Σ_a^A such that one of D's best responses to this strategy is x_j for some $j < \frac{n}{2} - a$. Then there is also a strategy in Σ_a^A such that all actions of the form x_k with $j \leq k \leq \frac{n}{2} - a$ are best responses for D.*

Proof. Consider some $j < \frac{n}{2} - a$ such that there is a strategy in Σ_a^A in which one of D's best responses to this strategy is x_j. Suppose by means of contradiction that there is no strategy in Σ_a^A such that all actions of the form x_k with $j \leq k \leq \frac{n}{2} - a$ are best responses for D. In that case, if A uses a strategy in Σ_a^A such that one of D's best responses is x_j, then there is some k satisfying $j < k \leq \frac{n}{2} - a$ such that x_k is not a best response for A.

Let h denote the unique integer satisfying $j < h \leq \frac{n}{2} - a$ such that A has a strategy in Σ_a^A for which all actions of the form x_k with $j \leq k < h$ are best responses for D but A does not have a strategy in Σ_a^A such that all actions of the form x_k with $j \leq k \leq h$ are best responses for D. Also let Σ_{ajh}^A denote the set of strategies for A in Σ_a^A such that all actions of the form x_k with $j \leq k < h$ are best responses for D. Finally, let $\pi \equiv \sup_{\sigma \in \Sigma_{ajh}^A} \Pi_D(\sigma, x_h)$ and for any $\sigma \in \Sigma_{ajh}^A$, let $\Pi \equiv \Pi_D(\sigma, x_j)$.

I claim that there is a strategy $\sigma \in \Sigma_{ajh}^A$ such that $\Pi_D(\sigma, x_h) = \pi$. To see this, let $\{\sigma^{(r)}\}_{r=1}^{\infty}$ denote an infinite sequence of strategies such that $\sigma^{(r)} \in \Sigma_{ajh}^A \; \forall \, r$ and $\lim_{r \to \infty} \Pi_D(\sigma^{(r)}, x_h) = \pi$. Note that $\sigma^{(r)} \in \Sigma_a \; \forall \, r$. So since Σ_a is a compact set, the infinite sequence $\{\sigma^{(r)}\}_{r=1}^{\infty}$ has a limit point in Σ_a.

Now let σ denote one of the limit points of the sequence $\{\sigma^{(r)}\}_{r=1}^{\infty}$ in Σ_a. I seek to demonstrate that $\sigma \in \Sigma_{ajh}^A$ and $\Pi_D(\sigma, x_h) = \pi$. First note that there is some subsequence of $\{\sigma^{(r)}\}_{r=1}^{\infty}$, say $\{\sigma^{(r_s)}\}_{s=1}^{\infty}$, such that $\lim_{s \to \infty} \sigma^{(r_s)} = \sigma$ since σ is a limit point of the sequence $\{\sigma^{(r)}\}_{r=1}^{\infty}$.

Now $\Pi_D(\sigma^{(r_s)}, x_k) = \Pi$ for all k satisfying $j \leq k < h$, $\lim_{s \to \infty} \Pi_D(\sigma^{(r_s)}, x_h) = \pi$, and $\Pi_D(\sigma^{(r_s)}, x_k) \leq \Pi$ for all positive integers k since each $\sigma^{(r_s)}$ is in Σ_{ajh}^A. But $\lim_{s \to \infty} \sigma^{(r_s)} = \sigma$ and $\Pi_D(\sigma, x_k)$ is continuous in σ for all k. Thus these facts imply that $\Pi_D(\sigma, x_k) = \Pi$ for all k satisfying $j \leq k < h$, $\Pi_D(\sigma, x_h) = \pi$, and $\Pi_D(\sigma, x_k) \leq \Pi$ for all positive integers k. And these facts together imply that $\sigma \in \Sigma_{ajh}^A$ and $\Pi_D(\sigma, x_h) = \pi$.

Since $\sigma \in \Sigma_{ajh}^A$, but there is no strategy in Σ_a^A such that all actions of the form x_k with $j \leq k \leq h$ are best responses for D, it must be the case that $\Pi_D(\sigma, x_h) < \Pi_D(\sigma, x_{h-1})$ and $\pi < \Pi$. To obtain a contradiction, it thus suffices to show that there is some $\sigma' \in \Sigma_{ajh}^A$ for which $\pi < \Pi_D(\sigma', x_h) \leq \Pi$, as this would contradict the definition of π.

First I show that $\sigma_{h+a-1} > 0$. To show this, note that it suffices to show that $\Pi_D(\sigma^i, x_h) \geq \Pi_D(\sigma^i, x_{h-1})$ for all positive integers i such that $i \leq \frac{n}{2}$ and $i \neq h+a-1$: For any symmetric strategy σ, we have $\sigma = \sum_{i=1}^{n/2} 2\sigma_i\sigma^i$ and $\Pi_D(\sigma, x_k) = \sum_{i=1}^{n/2} 2\sigma_i\Pi_D(\sigma^i, x_k)$ for all k. Thus if $\Pi_D(\sigma, x_h) < \Pi_D(\sigma, x_{h-1})$, then $\sum_{i=1}^{n/2} 2\sigma_i\Pi_D(\sigma^i, x_h) < \sum_{i=1}^{n/2} 2\sigma_i\Pi_D(\sigma^i, x_{h-1})$ and $2\sigma_{h+a-1}(\Pi_D(\sigma^{h+a-1}, x_{h-1}) - \Pi_D(\sigma^{h+a-1}, x_h)) > \sum_{i=1}^{h+a-2} 2\sigma_i(\Pi_D(\sigma^i, x_h) - \Pi_D(\sigma^i, x_{h-1})) + \sum_{i=h+a}^{n/2} 2\sigma_i(\Pi_D(\sigma^i, x_h) - \Pi_D(\sigma^i, x_{h-1}))$. But if $\Pi_D(\sigma^i, x_h) \geq \Pi_D(\sigma^i, x_{h-1})$ for all positive integers i such that $i \leq \frac{n}{2}$ and $i \neq h+a-1$, then this expression implies that $2\sigma_{h+a-1}(\Pi_D(\sigma^{h+a-1}, x_{h-1}) - \Pi_D(\sigma^{h+a-1}, x_h)) > 0$. Since this can only hold if $\sigma_{h+a-1} > 0$, it suffices to show that $\Pi_D(\sigma^i, x_h) \geq \Pi_D(\sigma^i, x_{h-1})$ for all positive integers i such that $i \leq \frac{n}{2}$ and $i \neq h+a-1$ in order to prove that $\sigma_{h+a-1} > 0$.

Now if $h+a \leq i \leq \frac{n}{2}$, then we know from Lemma 2(a) that $\Pi_D(\sigma^i, x_h) > \Pi_D(\sigma^i, x_{h-1})$. And if $1 \leq i \leq h+a-2$, then we know from Lemma 2(c) that $\Pi_D(\sigma^i, x_h) \geq \Pi_D(\sigma^i, x_{h-1})$. But this means that $\Pi_D(\sigma^i, x_h) \geq \Pi_D(\sigma^i, x_{h-1})$ for all positive integers i such that $i \leq \frac{n}{2}$ and $i \neq h+a-1$. From this it follows that $\sigma_{h+a-1} > 0$.

Now let σ^ϵ be the strategy defined by $\sigma^\epsilon \equiv \sigma - \epsilon\sigma^{h+a-1} + \epsilon\sigma^{h+a}$. Since $\sigma_{h+a-1} > 0$, for sufficiently small $\epsilon > 0$, we have $\sigma^\epsilon_{h+a-1} > 0$ and $\sigma^\epsilon \geq 0$. Thus σ^ϵ is a feasible symmetric strategy. To obtain a contradiction, it suffices to show that for sufficiently small $\epsilon > 0$, σ^ϵ is a strategy satisfying $\sigma^\epsilon \in \Sigma^A_{ajh}$ and $\Pi_D(\sigma^\epsilon, x_h) > \pi$. To prove this, it suffices to show that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all k if $\epsilon > 0$ is sufficiently small, $\Pi_D(\sigma^\epsilon, x_k) = \Pi_D(\sigma, x_k)$ for all k satisfying $j \leq k < h$, and $\Pi_D(\sigma^\epsilon, x_h) > \Pi_D(\sigma, x_h)$ if $\epsilon > 0$.

First I show that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all k if $\epsilon > 0$ is sufficiently small. To do this I first consider the case where $k \leq \frac{n}{2}$. Since $\Pi_D(\sigma, x_k)$ is continuous in σ for all k, the fact that $\Pi_D(\sigma, x_h) < \Pi$ means that $\Pi_D(\sigma^\epsilon, x_h) \leq \Pi$ for sufficiently small $\epsilon > 0$. Thus to prove that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all $k \leq \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small, it suffices to show that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi_D(\sigma, x_k)$ for all $k \leq \frac{n}{2}$ with $k \neq h$ if $\epsilon > 0$ is sufficiently small.

Now $\sigma^\epsilon - \sigma = \epsilon(\sigma^{h+a} - \sigma^{h+a-1})$. Thus for all k, $\Pi_D(\sigma^\epsilon, x_k) - \Pi_D(\sigma, x_k) = \epsilon(\Pi_D(\sigma^{h+a}, x_k) - \Pi_D(\sigma^{h+a-1}, x_k))$. So for $\epsilon > 0$, $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi_D(\sigma, x_k)$ holds if and only if $\Pi_D(\sigma^\epsilon, x_k) - \Pi_D(\sigma, x_k) \leq 0$, which holds if and only if $\Pi_D(\sigma^{h+a}, x_k) - \Pi_D(\sigma^{h+a-1}, x_k) \leq 0$ or $\Pi_A(\sigma^{h+a}, x_k) \geq \Pi_A(\sigma^{h+a-1}, x_k)$. So to prove that $\Pi_D(\sigma^\epsilon, x_k) \leq \Pi$ for all $k \leq \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small, it suffices to show that $\Pi_A(\sigma^{h+a}, x_k) \geq \Pi_A(\sigma^{h+a-1}, x_k)$ for all $k \leq \frac{n}{2}$ with $k \neq h$.

Now if $h < k \le \frac{n}{2}$, then we know from Lemma 1(c) that $\Pi_A(\sigma^{h+a}, x_k) \ge \Pi_A(\sigma^{h+a-1}, x_k)$. And if $k \le h-1$, then we know from Lemma 1(a) that $\Pi_A(\sigma^{h+a}, x_k) = \Pi_A(\sigma^{h+a-1}, x_k)$. Thus we have $\Pi_A(\sigma^{h+a}, x_k) \ge \Pi_A(\sigma^{h+a-1}, x_k)$ for all $k \le \frac{n}{2}$ with $k \ne h$, and $\Pi_D(\sigma^\epsilon, x_k) \le \Pi$ for all $k \le \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small.

But since σ^ϵ is a symmetric strategy, $\Pi_D(\sigma^\epsilon, x_k) = \Pi_D(\sigma^\epsilon, x_{n-k+1})$ for all k. Thus since $\Pi_D(\sigma^\epsilon, x_k) \le \Pi$ for all $k \le \frac{n}{2}$ if $\epsilon > 0$ is sufficiently small, we also have $\Pi_D(\sigma^\epsilon, x_k) \le \Pi$ for all k if $\epsilon > 0$ is sufficiently small.

Now I show that $\Pi_D(\sigma^\epsilon, x_k) = \Pi_D(\sigma, x_k)$ for all k satisfying $j \le k < h$. Since $\Pi_D(\sigma^\epsilon, x_k) - \Pi_D(\sigma, x_k) = \epsilon(\Pi_D(\sigma^{h+a}, x_k) - \Pi_D(\sigma^{h+a-1}, x_k))$ for all k, to prove that $\Pi_D(\sigma^\epsilon, x_k) = \Pi_D(\sigma, x_k)$ for all k satisfying $j \le k < h$, it suffices to show that $\Pi_D(\sigma^{h+a}, x_k) = \Pi_D(\sigma^{h+a-1}, x_k)$ for all k satisfying $j \le k < h$. But it follows immediately from Lemma 1(a) that $\Pi_D(\sigma^{h+a}, x_k) = \Pi_D(\sigma^{h+a-1}, x_k)$ for all k satisfying $j \le k < h$. Thus $\Pi_D(\sigma^\epsilon, x_k) = \Pi_D(\sigma, x_k)$ holds for all k satisfying $j \le k < h$.

To finish the proof, I only need to show that $\Pi_D(\sigma^\epsilon, x_h) > \Pi_D(\sigma, x_h)$ if $\epsilon > 0$. Since $\Pi_D(\sigma^\epsilon, x_h) - \Pi_D(\sigma, x_h) = \epsilon(\Pi_D(\sigma^{h+a}, x_h) - \Pi_D(\sigma^{h+a-1}, x_h))$, to prove that $\Pi_D(\sigma^\epsilon, x_h) > \Pi_D(\sigma, x_h)$ if $\epsilon > 0$, it suffices to show that $\Pi_D(\sigma^{h+a}, x_h) > \Pi_D(\sigma^{h+a-1}, x_h)$ or $\Pi_A(\sigma^{h+a}, x_h) < \Pi_A(\sigma^{h+a-1}, x_h)$. Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^{h+a}, x_h) < \Pi_A(\sigma^{h+a-1}, x_h)$ holds if and only if $\frac{1}{2}(\pi_A(x_{h+a}, x_h) + \pi_A(x_{n-h-a+1}, x_h)) < \frac{1}{2}(\pi_A(x_{h+a-1}, x_h) + \pi_A(x_{n-h-a+2}, x_h))$ or $\pi_A(x_{h+a-1}, x_h) - \pi_A(x_{h+a}, x_h) > \pi_A(x_{n-h-a+1}, x_h) - \pi_A(x_{n-h-a+2}, x_h)$. In order to prove that $\Pi_D(\sigma^\epsilon, x_h) > \Pi_D(\sigma, x_h)$ if $\epsilon > 0$, it thus suffices to show that $\pi_A(x_{h+a-1}, x_h) - \pi_A(x_{h+a}, x_h) > \frac{1}{n} \ge \pi_A(x_{n-h-a+1}, x_h) - \pi_A(x_{n-h-a+2}, x_h)$.

To see that $\pi_A(x_{h+a-1}, x_h) - \pi_A(x_{h+a}, x_h) > \frac{1}{n}$, note from equation (1) that $\pi_A(x_{h+a-1}, x_h) = 1$ and $\pi_A(x_{h+a}, x_h) = \frac{n-h}{n}$. Thus $\pi_A(x_{h+a-1}, x_h) - \pi_A(x_{h+a}, x_h) = \frac{h}{n}$. But x_{h-1} is a feasible policy in the policy space, so $h - 1 \ge 1$ and $h \ge 2$. Thus $\pi_A(x_{h+a-1}, x_h) - \pi_A(x_{h+a}, x_h) \ge \frac{2}{n} > \frac{1}{n}$.

To see that $\pi_A(x_{n-h-a+1}, x_h) - \pi_A(x_{n-h-a+2}, x_h) \le \frac{1}{n}$, first let $i = n-h-a+1$. Note that if $h = i-a-2j+1$ for any positive integer j, then from equation (1) it follows that $\pi_A(x_i, x_h) = \frac{n-i+a+j}{n}$, $\pi_A(x_{i+1}, x_h) = \frac{n-(i+1)+a+j}{n}$, and $\pi_A(x_{n-h-a+1}, x_h) - \pi_A(x_{n-h-a+2}, x_h) = \frac{1}{n}$. And if $h = i-a-2j+1$ does not hold for any positive integer j, then $h = i-a-2j$ for some nonnegative integer j and $h = i+1-a-2(j+1)+1$ for this same integer j. But then we see from equation (1) that $\pi_A(x_i, x_h) = \frac{n-i+a+j}{n}$ and $\pi_A(x_{i+1}, x_h) = \frac{n-(i+1)+a+j+1}{n} = \frac{n-i+a+j}{n}$, meaning $\pi_A(x_{n-h-a+1}, x_h) -$

$\pi_A(x_{n-h-a+2}, x_h) = 0$. In either case, we have $\pi_A(x_{n-h-a+1}, x_h) - \pi_A(x_{n-h-a+2}, x_h) \leq \frac{1}{n}$. From this it follows that $\Pi_D(\sigma^\epsilon, x_h) > \Pi_D(\sigma, x_h)$ if $\epsilon > 0$.

But then we see that for sufficiently small $\epsilon > 0$, σ^ϵ is a strategy in Σ^A_{ajh} for which $\Pi_D(\sigma^\epsilon, x_h) > \pi$. This contradicts the definition of π and proves the desired result.

□

Theorem 5. *If $\sigma \in \Sigma^A_a$, then one of D's best responses to σ is to choose an action of the form x_k for some positive integer $k \leq \frac{n}{2} - a$.*

Proof. Suppose by means of contradiction that $\sigma \in \Sigma^A_a$, but none of D's best responses to σ is to choose an action of the form x_k for some positive integer $k \leq \frac{n}{2} - a$. First I show that if one of D's best responses to σ is to choose $x_{n/2-a+1}$, then $\sigma_i > 0$ for some $i \leq \frac{n}{2} - 2$.

Since σ is a symmetric strategy, we have $\sigma = \sum_{i=1}^{n/2} 2\sigma_i\sigma^i$. Thus if $\sigma_i = 0$ for all $i \leq \frac{n}{2} - 2$, then $\sigma = 2\sigma_{n/2-1}\sigma^{n/2-1} + 2\sigma_{n/2}\sigma^{n/2}$ and $\Pi_D(\sigma, x_k) = 2\sigma_{n/2-1}\Pi_D(\sigma^{n/2-1}, x_k) + 2\sigma_{n/2}\Pi_D(\sigma^{n/2}, x_k)$. Now if $x_{n/2-a+1}$ is one of D's best responses to σ but D does not have a best response to σ that is of the form x_k for some positive integer $k \leq \frac{n}{2} - a$, then $\Pi_D(\sigma, x_{n/2-a+1}) > \Pi_D(\sigma, x_{n/2-a})$ or $\Pi_D(\sigma, x_{n/2-a+1}) - \Pi_D(\sigma, x_{n/2-a}) > 0$ or $2\sigma_{n/2-1}(\Pi_D(\sigma^{n/2-1}, x_{n/2-a+1}) - \Pi_D(\sigma^{n/2-1}, x_{n/2-a})) + 2\sigma_{n/2}(\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) - \Pi_D(\sigma^{n/2}, x_{n/2-a})) > 0$. This requires that either $\Pi_D(\sigma^{n/2-1}, x_{n/2-a+1}) > \Pi_D(\sigma^{n/2-1}, x_{n/2-a})$ or $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) > \Pi_D(\sigma^{n/2}, x_{n/2-a})$. Thus to show that $\sigma_i > 0$ for some $i \leq \frac{n}{2} - 2$ if one of D's best responses to σ is $x_{n/2-a+1}$, it suffices to show that $\Pi_D(\sigma^{n/2-1}, x_{n/2-a+1}) \leq \Pi_D(\sigma^{n/2-1}, x_{n/2-a})$ and $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) \leq \Pi_D(\sigma^{n/2}, x_{n/2-a})$. Proving this amounts to showing that $\Pi_A(\sigma^{n/2-1}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2-1}, x_{n/2-a})$ and $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2}, x_{n/2-a})$.

Recall that $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^i, x_{k+1}) \geq \Pi_A(\sigma^i, x_k)$ holds if and only if $\frac{1}{2}(\pi_A(x_i, x_{k+1}) + \pi_A(x_{n-i+1}, x_{k+1})) \geq \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$ or $\pi_A(x_i, x_{k+1}) - \pi_A(x_i, x_k) \geq \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i+1}, x_{k+1})$.

To show that $\Pi_A(\sigma^{n/2-1}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2-1}, x_{n/2-a})$, it thus suffices to show that $\pi_A(x_{n/2-1}, x_{n/2-a+1}) - \pi_A(x_{n/2-1}, x_{n/2-a}) \geq \pi_A(x_{n/2+2}, x_{n/2-a}) - \pi_A(x_{n/2+2}, x_{n/2-a+1})$. Now by equation (1) it follows that $\pi_A(x_{n/2+2}, x_{n/2-a}) = \pi_A(x_{n/2+2}, x_{n/2-a+1}) = \frac{n-(n/2+2)+a+1}{n}$ and $\pi_A(x_{n/2-1}, x_{n/2-a}) = \pi_A(x_{n/2-1}, x_{n/2-a+1}) = 1$. Thus $\pi_A(x_{n/2-1}, x_{n/2-a+1}) - \pi_A(x_{n/2-1}, x_{n/2-a}) \geq 0 \geq \pi_A(x_{n/2+2}, x_{n/2-a}) - \pi_A(x_{n/2+2}, x_{n/2-a+1})$ and we have $\Pi_A(\sigma^{n/2-1}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2-1}, x_{n/2-a})$.

And to show that $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2}, x_{n/2-a})$, it suffices to show that $\pi_A(x_{n/2}, x_{n/2-a+1}) - \pi_A(x_{n/2}, x_{n/2-a}) \geq \pi_A(x_{n/2+1}, x_{n/2-a}) - \pi_A(x_{n/2+1}, x_{n/2-a+1})$. By equation (1) it follows that

$\pi_A(x_{n/2+1}, x_{n/2-a}) = \frac{n-(n/2+1)+a+1}{n}$, $\pi_A(x_{n/2+1}, x_{n/2-a+1}) = \frac{n-(n/2+1)+a}{n}$, $\pi_A(x_{n/2}, x_{n/2-a}) = \frac{n-n/2+a}{n}$, and $\pi_A(x_{n/2}, x_{n/2-a+1}) = 1$. Thus $\pi_A(x_{n/2}, x_{n/2-a+1}) - \pi_A(x_{n/2}, x_{n/2-a}) = \frac{n/2-a}{n} \geq \frac{1}{n} = \pi_A(x_{n/2+1}, x_{n/2-a}) - \pi_A(x_{n/2+1}, x_{n/2-a+1})$ and we have $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2}, x_{n/2-a})$. Thus we have both $\Pi_A(\sigma^{n/2-1}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2-1}, x_{n/2-a})$ and $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) \geq \Pi_A(\sigma^{n/2}, x_{n/2-a})$, so $\sigma_i > 0$ for some $i \leq \frac{n}{2} - 2$ if one of D's best responses to σ is $x_{n/2-a+1}$.

Now let $\sigma^\epsilon = (1-\epsilon)\sigma + \epsilon\sigma^{n/2}$ for some arbitrarily small $\epsilon > 0$. Next I seek to show that if x_k is a best response to σ for some k satisfying $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$, then $\Pi_D(\sigma^\epsilon, x_k) < \Pi_D(\sigma, x_k)$ for any $\epsilon > 0$. To see this, first note that if x_k is a best response to σ, then since $x_{n/2-a}$ is not a best response, we have $\Pi_D(\sigma, x_k) > \Pi_D(\sigma, x_{n/2-a}) \geq 0$. Thus $\Pi_D(\sigma, x_k) > 0$ for any k satisfying $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$. Also, from equation (1) it follows that $\Pi_A(\sigma^{n/2}, x_k) = \frac{1}{2}(\pi_A(x_{n/2}, x_k) + \pi_A(x_{n/2+1}, x_k)) = 1$ for any k satisfying $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$. Thus $\Pi_D(\sigma^{n/2}, x_k) = 0$ for any such k. But $\Pi_D(\sigma^\epsilon, x_k) = (1-\epsilon)\Pi_D(\sigma, x_k) + \epsilon\Pi_D(\sigma^{n/2}, x_k)$ for any k. Thus if $\Pi_D(\sigma, x_k) > 0$ and $\Pi_D(\sigma^{n/2}, x_k) = 0$, then $\Pi_D(\sigma^\epsilon, x_k) < \Pi_D(\sigma, x_k)$ for any $\epsilon > 0$. From this we see that if x_k is a best response to σ for some k satisfying $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$, then $\Pi_D(\sigma^\epsilon, x_k) < \Pi_D(\sigma, x_k)$ for any $\epsilon > 0$.

Now I derive a contradiction to the assumption that $\sigma \in \Sigma_a^A$, but none of D's best responses to σ is to choose an action of the form x_k for some positive integer $k \leq \frac{n}{2} - a$. I consider two cases:

<u>Case 1</u>: Suppose $x_{n/2-a+1}$ is not a best response to σ for D. In this case, all actions of the form x_k for positive integers $k \leq \frac{n}{2} - a + 1$ are not best responses to σ for D. And σ is a symmetric strategy, so $\Pi_D(\sigma, x_k) = \Pi_D(\sigma, x_{n-k+1})$ for all k and all actions of the form x_{n-k+1} for positive integers $k \leq \frac{n}{2} - a + 1$ are not best responses to σ for D either. Thus any best response to σ for D must be of the form x_k for some k satisfying $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$.

Now let $\Pi \equiv \Pi_D(\sigma, x_k)$, where x_k is one of D's best responses to σ. I seek to show that if $\epsilon > 0$ is sufficiently small, then $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ for all k. In this case, σ would not be a maxminimizer strategy for A because $\sigma^A = \sigma^\epsilon$ would achieve a greater value of $\min_{\sigma^D} \Pi_A(\sigma^A, \sigma^D)$ than $\sigma^A = \sigma$.

To see this, first note that if x_k is a best response to σ, then $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$, and we know that $\Pi_D(\sigma^\epsilon, x_k) < \Pi_D(\sigma, x_k)$ for any $\epsilon > 0$. And if x_k is not a best response to σ, then $\Pi_D(\sigma, x_k) < \Pi$, and since Π_D is continuous in σ, $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ for sufficiently small $\epsilon > 0$. But this means that for sufficiently small $\epsilon > 0$, we have $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ for all k. Thus it cannot be the case that $x_{n/2-a+1}$ is not a best response to σ for D.

Case 2: Suppose $x_{n/2-a+1}$ is a best response to σ for D. In this case, we know that $\sigma_i > 0$ for some $i \leq \frac{n}{2} - 2$.

First note that $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) = \Pi_A(\sigma^{n/2-1}, x_{n/2-a+1}) > \Pi_A(\sigma^i, x_{n/2-a+1})$ for any $i \leq \frac{n}{2} - 2$. To see that $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) = \Pi_A(\sigma^{n/2-1}, x_{n/2-a+1})$, note that the fact that $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$ means that $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) = \Pi_A(\sigma^{n/2-1}, x_{n/2-a+1})$ holds if and only if $\frac{1}{2}(\pi_A(x_{n/2}, x_{n/2-a+1}) + \pi_A(x_{n/2+1}, x_{n/2-a+1})) = \frac{1}{2}(\pi_A(x_{n/2-1}, x_{n/2-a+1}) + \pi_A(x_{n/2+2}, x_{n/2-a+1}))$. But we know from equation (1) that $\pi_A(x_{n/2}, x_{n/2-a+1}) = \pi_A(x_{n/2-1}, x_{n/2-a+1}) = 1$, $\pi_A(x_{n/2+2}, x_{n/2-a+1}) = \frac{n-(n/2+2)+a+1}{n}$, and $\pi_A(x_{n/2+1}, x_{n/2-a+1}) = \frac{n-(n/2+1)+a}{n} = \pi_A(x_{n/2+2}, x_{n/2-a+1})$. Thus $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) = \Pi_A(\sigma^{n/2-1}, x_{n/2-a+1})$ indeed holds.

Now $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) > \Pi_A(\sigma^i, x_{n/2-a+1})$ holds for $i \leq \frac{n}{2} - 2$ if and only if $\frac{1}{2}(\pi_A(x_{n/2}, x_{n/2-a+1}) + \pi_A(x_{n/2+1}, x_{n/2-a+1})) > \frac{1}{2}(\pi_A(x_i, x_{n/2-a+1}) + \pi_A(x_{n-i+1}, x_{n/2-a+1}))$ or $\pi_A(x_{n/2}, x_{n/2-a+1}) - \pi_A(x_i, x_{n/2-a+1}) > \pi_A(x_{n-i+1}, x_{n/2-a+1}) - \pi_A(x_{n/2+1}, x_{n/2-a+1})$ holds for $i \leq \frac{n}{2} - 2$. But $\pi_A(x_{n/2}, x_{n/2-a+1}) = 1$, so $\pi_A(x_{n/2}, x_{n/2-a+1}) - \pi_A(x_i, x_{n/2-a+1}) \geq 0$. And if $i \leq \frac{n}{2} - 2$, then $n - i + 1 \geq \frac{n}{2} + 3$, and voters with ideal points no greater than $x_{n/2-a+2}$ all strictly prefer to vote for candidate D if $x_A = x_{n-i+1}$ and $x_D = x_{n/2-a+1}$. Thus $\pi_A(x_{n-i+1}, x_{n/2-a+1}) \leq \frac{n-(n/2+2)+a}{n}$ if $i \leq \frac{n}{2} - 2$. Combining this with the fact that $\pi_A(x_{n/2+1}, x_{n/2-a+1}) = \frac{n-(n/2+1)+a}{n}$ shows that $\pi_A(x_{n-i+1}, x_{n/2-a+1}) - \pi_A(x_{n/2+1}, x_{n/2-a+1}) \leq -\frac{1}{n}$ if $i \leq \frac{n}{2} - 2$. Thus $\pi_A(x_{n/2}, x_{n/2-a+1}) - \pi_A(x_i, x_{n/2-a+1}) \geq 0 > \pi_A(x_{n-i+1}, x_{n/2-a+1}) - \pi_A(x_{n/2+1}, x_{n/2-a+1})$ if $i \leq \frac{n}{2} - 2$, and $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) > \Pi_A(\sigma^i, x_{n/2-a+1})$ holds if $i \leq \frac{n}{2} - 2$.

Now let $\Pi \equiv \Pi_D(\sigma, x_{n/2-a+1})$. Since $\sigma = \sum_{i=1}^{n/2} 2\sigma_i \sigma^i$, we have $\Pi_D(\sigma, x_{n/2-a+1}) = \sum_{i=1}^{n/2} 2\sigma_i \Pi_D(\sigma^i, x_{n/2-a+1}) = 2(\sigma_{n/2-1} + \sigma_{n/2})\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) + \sum_{i=1}^{n/2-2} 2\sigma_i \Pi_D(\sigma^i, x_{n/2-a+1})$. Combining this with the facts that $\Pi_A(\sigma^{n/2}, x_{n/2-a+1}) > \Pi_A(\sigma^i, x_{n/2-a+1})$ (and thus $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) < \Pi_D(\sigma^i, x_{n/2-a+1})$) holds for $i \leq \frac{n}{2} - 2$, and $\sigma_i > 0$ for some $i \leq \frac{n}{2} - 2$ means that $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) < \Pi$: If we had $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) \geq \Pi$, then we would have $\Pi_D(\sigma^i, x_{n/2-a+1}) > \Pi$ for any $i \leq \frac{n}{2} - 2$, which would in turn mean that $2(\sigma_{n/2-1} + \sigma_{n/2})\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) + \sum_{i=1}^{n/2-2} 2\sigma_i \Pi_D(\sigma^i, x_{n/2-a+1}) > \Pi$, contradicting the fact that $\Pi = \Pi_D(\sigma, x_{n/2-a+1}) = 2(\sigma_{n/2-1} + \sigma_{n/2})\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) + \sum_{i=1}^{n/2-2} 2\sigma_i \Pi_D(\sigma^i, x_{n/2-a+1})$. Thus $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) < \Pi$.

I now seek to show that if $\epsilon > 0$ is sufficiently small, then $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ for all k. First I consider the cases $k = \frac{n}{2} - a + 1$ and $k = \frac{n}{2} + a$. Since $\Pi_D(\sigma^\epsilon, x_{n/2-a+1}) = (1 - \epsilon)\Pi_D(\sigma, x_{n/2-a+1}) + \epsilon \Pi_D(\sigma^{n/2}, x_{n/2-a+1})$ and $\Pi_D(\sigma^{n/2}, x_{n/2-a+1}) < \Pi = \Pi_D(\sigma, x_{n/2-a+1})$, it follows that $\Pi_D(\sigma^\epsilon, x_{n/2-a+1}) <$

$\Pi_D(\sigma, x_{n/2-a+1}) = \Pi$ for any $\epsilon > 0$. And since σ^ϵ is a symmetric strategy, $\Pi_D(\sigma^\epsilon, x_{n/2+a}) = \Pi_D(\sigma^\epsilon, x_{n/2-a+1}) < \Pi$ for any $\epsilon > 0$ as well.

Now if $k \leq \frac{n}{2} - a$ or $k \geq \frac{n}{2} + a + 1$, then x_k is not a best response to σ for D and $\Pi_D(\sigma, x_k) < \Pi$. Thus since $\Pi_D(\sigma, x_k)$ is continuous in σ, if $\epsilon > 0$ is sufficiently small, we have $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ as well.

Finally, if $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$, then we have $\Pi_D(\sigma^{n/2}, x_k) = 0$. Thus since $\Pi_D(\sigma^\epsilon, x_k) = (1-\epsilon)\Pi_D(\sigma, x_k) + \epsilon \Pi_D(\sigma^{n/2}, x_k)$, if $\epsilon > 0$, we either have $\Pi_D(\sigma^\epsilon, x_k) = 0$ or $\Pi_D(\sigma^\epsilon, x_k) < \Pi_D(\sigma, x_k)$. In either case, we have $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ if $\epsilon > 0$. Thus for sufficiently small $\epsilon > 0$, we have $\Pi_D(\sigma^\epsilon, x_k) < \Pi$ for all k.

Thus regardless of whether we are in Case 1 or Case 2, we see that for sufficiently small $\epsilon > 0$, $\sigma^A = \sigma^\epsilon$ would achieve a greater value of $\min_{\sigma^D} \Pi_A(\sigma^A, \sigma^D)$ than $\sigma^A = \sigma$. This contradicts the assumption that σ is a maxminimizer strategy and proves the desired result.

□

Theorem 6. *There is an equilibrium (σ^A, σ^D) in symmetric strategies characterized by two positive integers k_A and k_D satisfying $k_D \leq \frac{n}{2} - a$ and $a \leq k_A \leq \frac{n}{2}$ such that the following hold:*

(a). All actions of the form x_i with $k_A \leq i \leq n - k_A + 1$ are best responses for A to D's strategy.

(b). $\sigma_i^A = 0$ if $i < k_A$ or $i > n - k_A + 1$.

(c). All actions of the form x_k with $k_D \leq k \leq \frac{n}{2} - a$ and $\frac{n}{2} + a + 1 \leq k \leq n - k_D + 1$ are best responses for D to A's strategy.

(d). No actions of the form x_k with $k < k_D$ or $k > n - k_D + 1$ are best responses for D to A's strategy.

(e). $k_D + a - 2 \leq k_A \leq k_D + a$.

Proof. From Theorem 1, we know that there is an equilibrium in symmetric strategies. We also know from Theorems 4 and 5 that A has an equilibrium strategy in $\sigma^A \in \Sigma_a^A$ such that there is some $k_D \leq \frac{n}{2} - a$ for which all actions of the form x_k with $k_D \leq k \leq \frac{n}{2} - a$ are best responses for D and all actions of the form x_k with $k < k_D$ are not best responses for D. By the symmetry of D's payoff function, for any such strategy, all actions of the form x_k with $\frac{n}{2} + a + 1 \leq k \leq n - k_D + 1$ are best responses for D, and all actions of the form x_k with $k > n - k_D + 1$ are not best responses for D. Thus there is an equilibrium strategy for A which satisfies parts (c) and (d) of the theorem.

Now let k_A be the smallest positive integer such that there is some $\sigma \in \Sigma_a^A$ with $\sigma_{k_A} > 0$. We know from Theorem 3 that if D uses some symmetric strategy σ^D such that (σ^A, σ^D) is an equilibrium, then all actions of the form x_i with $k_A \leq i \leq n - k_A + 1$ are best responses for A to D's strategy. And since $\sigma^A \in \Sigma_a^A$ and k_A is the smallest positive integer such that there is some $\sigma \in \Sigma_a^A$ with $\sigma_{k_A} > 0$, then we also know that $\sigma_i^A = 0$ if $i < k_A$ or $i > n - k_A + 1$. Thus we know there is some equilibrium (σ^A, σ^D) in symmetric strategies that satisfies parts (a)-(d) of the theorem.

So to prove the theorem it suffices to show that $k_D + a - 2 \leq k_A \leq k_D + a$. To see that $k_A \leq k_D + a$, first note that there is nothing to prove if $k_D = \frac{n}{2} - a$ since $k_A \leq \frac{n}{2}$ by definition. So suppose that $k_D < \frac{n}{2} - a$ and let i be a positive integer satisfying $k_D + a + 1 \leq i \leq \frac{n}{2}$. Note from Lemma 2(a) that $\Pi_D(\sigma^i, x_{k_D+1}) > \Pi_D(\sigma^i, x_{k_D})$ for any such i.

Since σ^A is a symmetric strategy, we have $\sigma^A = \sum_{i=1}^{n/2} 2\sigma_i^A \sigma^i$. Furthermore, since $\sigma_i^A = 0$ if $i < k_A$, we have $\sigma^A = \sum_{i=k_A}^{n/2} 2\sigma_i^A \sigma^i$. Thus $\Pi_D(\sigma^A, x_k) = \sum_{i=k_A}^{n/2} 2\sigma_i^A \Pi_D(\sigma^i, x_k)$ for all k. But then if $k_A \geq k_D + a + 1$, the result from the previous paragraph implies that $\Pi_D(\sigma^A, x_{k_D+1}) > \Pi_D(\sigma^A, x_{k_D})$. This contradicts the fact that x_{k_D} is a best response for D to A's strategy, so it must be the case that $k_A \leq k_D + a$.

Now I show that $k_D + a - 2 \leq k_A$. To see this, I first show that if $i = k_D + a - 3$, then σ^i is not a best response for A to D's strategy. In particular, I show that σ^{i+2} affords A a strictly larger expected payoff than σ^i against at least one action which D takes with positive probability, and σ^{i+2} also affords A an expected payoff at least as large as σ^i against every action D takes with positive probability.

To see this, first note that D must take some action x_k with $k \leq \frac{n}{2} - a + 1$ with positive probability. If the only actions D takes with positive probability are of the form x_k with $k \geq \frac{n}{2} - a + 2$, then the fact that D is using a symmetric strategy means that the only actions D takes with positive probability are of the form x_k with $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2} + a - 1$. But in this case, A could win with probability one by choosing the action $x_{n/2}$. This contradicts the fact that D is using a maxminimizing strategy because if D were using the strategy $\sigma^{n/2-a}$, there would be no action A could take that would win with probability one. Thus D must take some action x_k with $k \leq \frac{n}{2} - a + 1$ with positive probability.

Now if D takes an action with positive probability, then the action must be of the form x_k or x_{n-k+1} with $k_D \leq k \leq \frac{n}{2}$. And A's expected payoff from using a strategy of the form σ^i is the same regardless of whether D uses the action x_k or x_{n-k+1}. Thus to show σ^{i+2} affords A a strictly larger expected payoff than σ^i against D's strategy, we know from the previous two paragraphs that it suffices to show that $\Pi_A(\sigma^{i+2}, x_k) \geq \Pi_A(\sigma^i, x_k)$ for any positive integer k satisfying $\frac{n}{2}-a+2 \leq k \leq \frac{n}{2}$ and $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ for any positive integer k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$.

If $k \geq \frac{n}{2} - a + 2$, then $k + a - 1 \geq \frac{n}{2} + 1$. And since $i = k_D + a - 3$ and $k_D \leq \frac{n}{2} - a$, we have $i \leq \frac{n}{2} - 3$ and $i + 2 \leq \frac{n}{2} - 1$. Thus if $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2}$, we have $i + 2 \leq k + a - 1$ and we know from Lemma 1(c) that $\Pi_A(\sigma^{i+2}, x_k) \geq \Pi_A(\sigma^i, x_k)$. So to prove the result, it suffices to show that $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ for any positive integer k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$.

Note that if $i \leq k - a$, then we know from Lemma 1(b) that $\Pi_A(\sigma^{i+1}, x_k) > \Pi_A(\sigma^i, x_k)$ and we know from Lemma 1(c) that $\Pi_A(\sigma^{i+2}, x_k) \geq \Pi_A(\sigma^{i+1}, x_k)$. Thus $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ if $i \leq k - a$. So to prove that $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ for any positive integer k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$, it suffices to show that $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ if $k_D \leq k \leq \frac{n}{2} - a + 1$ and $i \geq k - a + 1$.

Now $\Pi_A(\sigma^i, x_k) = \frac{1}{2}(\pi_A(x_i, x_k)+\pi_A(x_{n-i+1}, x_k))$. Thus $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ holds if and only if $\frac{1}{2}(\pi_A(x_{i+2}, x_k) + \pi_A(x_{n-i-1}, x_k)) > \frac{1}{2}(\pi_A(x_i, x_k) + \pi_A(x_{n-i+1}, x_k))$ or $\pi_A(x_{i+2}, x_k) - \pi_A(x_i, x_k) > \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i-1}, x_k)$. It thus suffices to show that $\pi_A(x_{i+2}, x_k) - \pi_A(x_i, x_k) \geq 0 > \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i-1}, x_k)$ for all positive integers k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$ and $i \geq k - a + 1$.

Since $i \leq \frac{n}{2}-3$, we have $n-i+1 \geq \frac{n}{2}+4$. Thus the fact that $k \leq \frac{n}{2}-a+1$ implies that $k \leq (n-i+1)-a-3$. From this it follows that either $k = (n-i+1)-a-2j$ or $k = (n-i+1)-a-2j+1$ for some positive integer $j \geq 2$. We then have either $k = (n-i-1)-a-2(j-1)$ or $k = (n-i-1)-a-2(j-1)+1$ for this same integer j. Thus it follows from equation (1) that $\pi_A(x_{n-i+1}, x_k) = \frac{n-(n-i+1)+a+j}{n}$ for some positive integer $j \geq 2$ and $\pi_A(x_{n-i-1}, x_k) = \frac{n-(n-i-1)+a+j-1}{n}$ for this same integer j. But this means that $\pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i-1}, x_k) = -\frac{1}{n}$. Thus $0 > \pi_A(x_{n-i+1}, x_k) - \pi_A(x_{n-i-1}, x_k)$ for all positive integers k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$.

Now I show that $\pi_A(x_{i+2}, x_k) - \pi_A(x_i, x_k) \geq 0$ for all positive integers k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$ and $i \geq k - a + 1$. Since $i + 2 = k_D + a - 1$ and $k \geq k_D$, we have $k \geq i + 2 - a + 1$. And $i \geq k-a+1$ implies $k \leq i+2+a-1$. Thus we have $i+2-a+1 \leq k \leq i+2+a-1$, and we know from equation (1) that $\pi_A(x_{i+2}, x_k) = 1$. But this immediately implies that $\pi_A(x_{i+2}, x_k) - \pi_A(x_i, x_k) \geq 0$

for all positive integers k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$ and $i \geq k - a + 1$. From this it follows that $\Pi_A(\sigma^{i+2}, x_k) > \Pi_A(\sigma^i, x_k)$ for any positive integer k satisfying $k_D \leq k \leq \frac{n}{2} - a + 1$.

This means that if $i = k_D + a - 3$, then σ^{i+2} affords A a strictly larger expected payoff than σ^i against D's strategy and σ^i is not a best response for A to D's strategy. And since D is using a symmetric strategy, A's payoff is the same regardless of whether A takes the action x_i or x_{n-i+1}. Thus since σ^i, a strategy which mixes between x_i and x_{n-i+1}, is not a best response for A to D's strategy, x_i is not a best response for A to D's strategy either. But this means that we cannot have $k_A \leq i$. From this it follows that $k_A \geq k_D + a - 2$.

□

Theorem 7. $\limsup_{n\to\infty} \underline{x}(n, \delta) \leq \max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}$.

Proof. Note that $\limsup_{n\to\infty} \underline{x}(n, \delta) \leq \frac{1}{2}$ because any strategy $\sigma \in \Sigma^A$ has $\sigma_i > 0$ for some $i \leq \frac{n}{2}$ by definition of a symmetric strategy. Thus $\underline{x}(n, \delta) \leq x_{n/2} < \frac{1}{2}$ for all n and $\limsup_{n\to\infty} \underline{x}(n, \delta) \leq \frac{1}{2}$. So to prove that $\limsup_{n\to\infty} \underline{x}(n, \delta) \leq \max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}$, it suffices to show that $\limsup_{n\to\infty} \underline{x}(n, \delta) \neq \Delta$ for all $\Delta \in (\max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}, \frac{1}{2}]$.

To see this, suppose by means of contradiction that $\limsup_{n\to\infty} x(n, \delta) = \Delta$ for some $\Delta \in (\max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}, \frac{1}{2}]$. Consider some sufficiently small $\epsilon > 0$ and some sufficiently large integer N such that $\Delta - \epsilon - \frac{2}{N} > \max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}$. Since $\limsup_{n\to\infty} \underline{x}(n, \delta) = \Delta$, we know there exists some $n > N$ for which there is some $\sigma^A \in \Sigma^A$ satisfying $\sigma^A_i = 0$ for all i such that $x_i < \Delta - \epsilon$. Since σ^A is a symmetric strategy, this implies that $\sigma^A_i = 0$ for all i satisfying $x_i \notin [\Delta - \epsilon, 1 - \Delta + \epsilon]$.

For this n, let x_j denote the most liberal policy in X that is also in $[\Delta - \epsilon, 1 - \Delta + \epsilon]$ and suppose that A is using the strategy σ^A given in the previous paragraph. I seek to demonstrate that this implies that any best response for D, x_k, satisfies either $x_k \in [x_{j-a}, x_{n/2-a}]$ or $x_k \in [x_{n/2+1+a}, x_{n-j+1+a}]$.

I first note that $\frac{j}{n} > \frac{n+a+2}{3n}$ and $\frac{j}{n} > \frac{n-2a+2}{2n}$. Since x_j is in $[\Delta - \epsilon, 1 - \Delta + \epsilon]$, we have $x_j = \frac{j-1}{n-1} \geq \Delta - \epsilon$. And since $j < n$, this implies that $\frac{j}{n} > \frac{j-1}{n-1} \geq \Delta - \epsilon$. Now since $\Delta - \epsilon - \frac{2}{N} > \max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}$, we have $\Delta - \epsilon - \frac{1}{n} > \frac{1+\delta}{3} > \frac{n}{3n} + \frac{a-1}{3(n-1)} > \frac{n}{3n} + \frac{a-1}{3n}$ and $\Delta - \epsilon > \frac{n+a+2}{3n}$. And we also have $\Delta - \epsilon - \frac{2}{n} > \frac{1}{2} - \delta > \frac{n}{2n} - \frac{a}{n-1} \geq \frac{n}{2n} - \frac{a+1}{n} = \frac{n-2a-2}{2n}$ and $\Delta - \epsilon > \frac{n-2a+2}{2n}$. Thus we have $\frac{j}{n} > \frac{n+a+2}{3n}$ and $\frac{j}{n} > \frac{n-2a+2}{2n}$.

Since σ^A is a symmetric strategy, we have $\sigma^A = \sum_{i=1}^{n/2} 2\sigma^A_i \sigma^i$. Furthermore, since $\sigma^A_i = 0$ if $i < j$, we have $\sigma^A = \sum_{i=j}^{n/2} 2\sigma^A_i \sigma^i$. Thus $\Pi_D(\sigma^A, x_k) = \sum_{i=j}^{n/2} 2\sigma^A_i \Pi_D(\sigma^i, x_k)$ for all k. But if D chooses some action $x_k < x_{j-a}$, then we know from Lemma 2(a) that $\Pi_D(\sigma^i, x_{k+1}) > \Pi_D(\sigma^i, x_k)$ for all i

satisfying $j \leq i \leq \frac{n}{2}$. Thus if D chooses some action $x_k < x_{j-a}$, then $\Pi_D(\sigma^A, x_{k+1}) > \Pi_D(\sigma^A, x_k)$, and x_k is not a best response for D to σ^A.

Now note that if D chooses the action x_{j-a} when A uses the strategy σ^A, then D wins with probability at least $\frac{j-a}{n}$ because voters with ideal points $x_i \leq x_{j-a}$ always prefer candidate D to candidate A. By contrast, if D chooses some action $x_k \in [x_{n/2-a+1}, x_{n/2}]$, then D loses with probability 1 whenever A chooses some action $x_i \in [x_j, x_{n/2}]$: Since $\frac{j}{n} > \frac{n-2a+2}{2n}$, we have $j > \frac{n}{2} - a + 1$, and all actions in $[x_j, x_{n/2}]$ are within $a-1$ grid points of all actions in $[x_{n/2-a+1}, x_{n/2}]$. Furthermore, if D chooses some action $x_k \in [x_{n/2-a+1}, x_{n/2}]$ and A chooses some action $x_i \in [x_{n/2+1}, x_{n-j+1}]$, then D wins with probability no greater than $\frac{n-j-a+1}{n}$ since A wins whenever the median voter has ideal point $x_m \geq x_{n-j-a+2}$. Thus if D chooses some action $x_k \in [x_{n/2-a+1}, x_{n/2}]$, then D wins with probability no greater than $\frac{n-j-a+1}{2n}$ if A uses the strategy σ^A.

From this we see that if $\frac{j-a}{n} > \frac{n-j-a+1}{2n}$ or $3j > n + a + 1$ or $j > \frac{n+a+1}{3}$, then D obtains a greater payoff by choosing the action x_{j-a} than by choosing some action $x_k \in [x_{n/2-a+1}, x_{n/2}]$. But we have seen that $\frac{j}{n} > \frac{n+a+2}{3n}$ and thus $j > \frac{n+a+1}{3}$. Thus it is not a best response for D to choose some action $x_k \in [x_{n/2-a+1}, x_{n/2}]$ when A uses the strategy σ^A. From this it follows that if $x_k \leq x_{n/2}$ is a best response for D to σ^A, then $x_k \in [x_{j-a}, x_{n/2-a}]$. Similar reasoning shows that if $x_k \geq x_{n/2+1}$ is a best response for D to σ^A, then $x_k \in [x_{n/2+1+a}, x_{n-j+1+a}]$.

Thus if A uses the strategy σ^A, then any best response for D will only choose actions in either $[x_{j-a}, x_{n/2-a}]$ or $[x_{n/2+1+a}, x_{n-j+1+a}]$ with positive probability. Now I seek to show that if D is using some such strategy, σ^D, then A could obtain a higher payoff by choosing some action from x_{j-1} or x_{n-j+2} than by choosing the action $x_{n/2}$.

Note that if A chooses the action $x_{n/2}$ and D chooses some action in $[x_{j-a}, x_{n/2-a}]$, then D wins with probability at least $\frac{j-a}{n}$ because voters with ideal points $x_i \leq x_{j-a}$ always prefer candidate D to candidate A. And if A chooses the action $x_{n/2}$ and D chooses some action in $[x_{n/2+1+a}, x_{n-j+1+a}]$, then D wins with probability at least $\frac{j-a}{n}$ because voters with ideal points $x_i \geq x_{n-j+1+a}$ always prefer candidate D to candidate A. Thus if A chooses the action $x_{n/2}$ and D uses the strategy σ^D, then A wins with probability no greater than $\frac{n-j+a}{n}$.

By contrast, if A chooses the action x_{j-1} and D chooses some action in $[x_{j-a}, x_{n/2-a}]$, then A wins with probability 1: Since $\frac{j}{n} > \frac{n-2a+2}{2n}$, we have $j > \frac{n}{2} - a + 1$ and $j - 1 > \frac{n}{2} - a$, so x_{j-1} is within $a - 1$ grid points of all policies in $[x_{j-a}, x_{n/2-a}]$. And if A chooses the action x_{j-1} and D

chooses some action in $[x_{n/2+1+a}, x_{n-j+1+a}]$, then A wins with probability at least $\frac{j+a-2}{n}$ because voters with ideal points $x_i \leq x_{j+a-2}$ always prefer candidate A to candidate D. Thus if π denotes the probability with which σ^D chooses an action in $[x_{j-a}, x_{n/2-a}]$, then A wins with probability at least $\pi + (1-\pi)\frac{j+a-2}{n}$ if A chooses the action x_{j-1} and D uses the strategy σ^D. Similar reasoning shows that A wins with probability at least $1 - \pi + \pi(\frac{j+a-2}{n})$ if A chooses the action x_{n-j+2} and D uses the strategy σ^D.

Now if $\pi \geq \frac{1}{2}$, then $\pi + (1-\pi)\frac{j+a-2}{n} \geq \frac{n+j+a-2}{2n}$. And if $\pi < \frac{1}{2}$, then $1 - \pi + \pi(\frac{j+a-2}{n}) \geq \frac{n+j+a-2}{2n}$. Thus if D uses the strategy σ^D, then A can win with probability at least $\frac{n+j+a-2}{2n}$ by choosing some action from x_{j-1} or x_{n-j+2}. Thus if $\frac{n+j+a-2}{2n} > \frac{n-j+a}{n}$ or $3j > n+a+2$ or $j > \frac{n+a+2}{3}$, then A can win with greater probability by choosing some action from x_{j-1} or x_{n-j+2} than by choosing the action $x_{n/2}$. But since $\frac{j}{n} > \frac{n+a+2}{3n}$, we have $j > \frac{n+a+2}{3}$. Thus $x_{n/2}$ is not a best response for A to σ^D.

Now we know from Theorem 3 that if (σ^A, σ^D) is an equilibrium, then $x_{n/2}$ must be a best response for A to σ^D. Thus from the previous paragraph we know that (σ^A, σ^D) is not an equilibrium if σ^D is a strategy which only chooses actions in either $[x_{j-a}, x_{n/2-a}]$ or $[x_{n/2+1+a}, x_{n-j+1+a}]$ with positive probability. But we have also seen that any best response for D to σ^A must only choose actions in $[x_{j-a}, x_{n/2-a}]$ or $[x_{n/2+1+a}, x_{n-j+1+a}]$ with positive probability. Thus there is no equilibrium in which A uses the strategy σ^A, and $\limsup_{n\to\infty} \underline{x}(n,\delta) = \Delta$ cannot hold for any $\Delta \in (\max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}, \frac{1}{2}]$. Thus $\limsup_{n\to\infty} \underline{x}(n,\delta) \leq \max\{\frac{1+\delta}{3}, \frac{1}{2} - \delta\}$.

$\square$

Theorem 8. *The probability the advantaged candidate wins the election is at least $\frac{1}{2} + \frac{a}{n}$.*

Proof. Suppose A uses the strategy $\sigma^{n/2}$. Note that if A uses this strategy, then the action $x_{n/2-a}$ is a best response for D to A's strategy: We know from Lemma 2(a) that D's expected payoff from taking the action $x_{n/2-a}$ is strictly greater than D's expected payoff from taking the action x_k for all $k < \frac{n}{2} - a$. We also know that if D takes an action x_k with $\frac{n}{2} - a + 2 \leq k \leq \frac{n}{2}$, then D loses with certainty. Finally, if D takes the action $x_{n/2-a+1}$, then D's expected payoff is $\frac{1}{2}(\pi_D(x_{n/2}, x_{n/2-a+1}) + \pi_D(x_{n/2+1}, x_{n/2-a+1})) = \frac{1}{2}(0 + \frac{n/2-a+1}{n}) = \frac{n/2-a+1}{2n}$. But if D takes the action $x_{n/2-a}$, then D's expected payoff is $\frac{1}{2}(\pi_D(x_{n/2}, x_{n/2-a}) + \pi_D(x_{n/2+1}, x_{n/2-a})) = \frac{1}{2}(\frac{n/2-a}{n} + \frac{n/2-a}{n}) = \frac{n/2-a}{n}$. And since $\frac{n}{2} - a \geq 1$, we have $\frac{n}{2} - a + 1 \leq 2(\frac{n}{2} - a)$ and $\frac{n/2-a+1}{2n} \leq \frac{n/2-a}{n}$. Thus D's expected payoff from taking the action $x_{n/2-a}$ is at least as large as D's expected payoff from taking the action $x_{n/2-a+1}$.

From this it follows that D's expected payoff from taking the action $x_{n/2-a}$ is at least as large as D's expected payoff from taking any other action x_k with $k \leq \frac{n}{2}$. But we know that D's expected payoff from taking an action x_k with $k \leq \frac{n}{2}$ is the same as D's expected payoff from taking an x_{n-k+1} with $k \leq \frac{n}{2}$. Thus it follows that D's expected payoff from taking the action $x_{n/2-a}$ is at least as high as D's expected payoff from taking any other action x_k.

Now we have seen in the first paragraph that D's expected payoff from taking the action $x_{n/2-a}$ is $\frac{n/2-a}{n}$. Thus if A uses the strategy $\sigma^{n/2}$, then D wins with probability no greater than $\frac{n/2-a}{n}$. But this means that A can guarantee that A will win with probability equal to at least $1 - \frac{n/2-a}{n} = \frac{1}{2} + \frac{a}{n}$ by using the strategy $\sigma^{n/2}$. Thus the probability the advantaged candidate wins the election is at least $\frac{1}{2} + \frac{a}{n}$.

$\square$

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