Statistical Game Theory *

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Abstract

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JEL classification: C72 Noncooperative Games

Keywords: correlated equilibrium, identification, spatial data analysis, local interaction games

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1. *Introduction*

Game theory usually predicts a "point," such as Nash equilibrium, or a set, such as the von Neumann-Morgenstern solution. This paper shows how game theory can make another kind of prediction: a statistical relationship between people's strategies. I show that in any correlated equilibrium of a game, when a person chooses among two strategies, conditional on these two strategies, her strategy is nonnegatively correlated with the payoff difference between the two strategies. This result has several implications. In a two person game in which one person's payoffs are linear-quadratic, for example, one can predict the sign of the covariance of people's strategies. In a local interaction game, one can predict the sign of the covariance between a person's strategy and the number of neighbors who play the same strategy. This paper also considers the question of identification: given observations, what games are consistent with these observations. For 2×2 games, for example, a signed covariance in people's strategies is sufficient to identify pure strategy Nash equilibria of the game.

The main motivation for this "statistical approach" is simply that most real world data display variation. Correlated equilibria (Aumann 1974) allow statistical relationships to be derived directly from a standard strategic form game. The correlated equilibrium approach has several advantages. First, the set of correlated equilibria of a given game is convex; hence any aggregation of correlated equilibria is also a correlated equilibrium (this is not true for Nash equilibrium, for example). One need not worry about whether a given observation is an individual or aggregate, since they can be treated in the same way. For example, if each group of individuals in a population is playing a correlated equilibrium, or if each play in a sequence of trials is a correlated equilibrium, then the distribution of actions aggregated over the population or over the sequence is a correlated equilibrium.

Second, the set of correlated equilibria contains all convex combinations of all pure strategy and mixed strategy Nash equilibria. For example, one might not know whether a given observed strategy profile is one of several pure strategy Nash equilibria, or one realization of a probability distribution which is one of several mixed strategy Nash equilibria. Aggregating over many plays, one will end up with a mixture of pure strategy Nash equilibria and mixed strategy Nash equilibrium. This mixture will still be a correlated equilibrium, however. The assumption of correlated equilibrium is logically weaker than the assumption of Nash equilibrium and hence any results are logically stronger. If one is more comfortable with assuming that observed behavior is a mixture of Nash equilibria, all of the results in this paper still hold because they are derived under weaker assumptions.

Third, the set of correlated equilibria includes all possible equilibrium behavior for all possible communication and coordination mechanisms (see for example Myerson 1991). Hence any communication technology, implicit or explicit, or none at all, can be allowed without having to specify any mechanism in particular. For example, each person might condition her action on some random variable, and these random variables, unknown to the outside observer, might be correlated; correlated equilibrium handles this and all other possibilities naturally.

Fourth, the issue of multiple equilibria is a familiar one for game theory, and many arguments have been developed to justify some equilibria over others. The approach here completely avoids this issue by considering all correlated equilibria of a game. Here we are concerned not with any single predicted action but statistical relationships among actions; a wide range of possible actions is something to be embraced, not avoided.

Finally, most approaches toward explaining observed statistical variation in a game have been based on Nash equilibria. Because Nash equilibrium is a point prediction, explaining observed variation requires adding exogenous randomness or heterogeneity, for example by allowing random mistakes or by allowing payoffs in a game to vary randomly (for example McKelvey and Palfrey 1995, Lewis and Schultz 2003, and Signorino 2003). Because games often have multiple Nash equilibria, the distribution of observed outcomes in such a model is not exactly determined by assumptions over how the game randomly varies. Also, often games have mixed strategy Nash equilibria, and hence there is an endogenous source of randomness and variation along with the exogenous randomness and variation added to the game. These complications, serious enough to be considered research questions (for example Bresnahan and Weiss 1991, Tamer 2003), are completely avoided by the more basic correlated equilibrium approach. Statistical relationships can be derived directly from an unadorned strategic form game. A more elaborate model which involves several assumptions not related to the game itself is not necessary. Of course it would be good to add "errors" and exogenous randomness to the correlated equilibrium approach, but these questions are logically separate from the basic questions of identification and prediction (see for example Manski 1995). The first step is to consider the "pure" case without exogenous randomness.

Although correlated equilibrium can be understood as a more fundamental concept than Nash equilibrium (as argued by Aumann 1987), it is not as commonly used. This paper tries to remedy two possible reasons for this. The first is that despite its mathematical simplicity, correlated equilibrium is sometimes seen as complicated, requiring for example the construction of a complicated and artificial messaging device. This paper interprets correlated equilibrium in simple statistical terms as a histogram of observed strategy profiles (again, assuming no "errors") which satisfies individual rationality. The second is that even in simple games, the set of correlated equilibria of a game can be hard to "visualize" intuitively (see Nau, Gomez-Canovas, and Hansen 2003 and Calvó-Armengol 2003). This paper shows how correlated equilibria can be described in "reduced form" in terms of a signed covariance, a basic statistical concept.

2. *Definitions*

We have a standard finite strategic form game: a finite set of players $N = \{1, \ldots, n\}$, with each person $i \in N$ choosing an action a_i from the finite set A_i, and with each person $i \in N$ having a payoff function $u_i : A \rightarrow \Re$, where $A = \times_{i \in N} A_i$. We assume that $A_i \subset \Re$; each strategy a_i is a real number. We write a game as u, where $u = (u_1, \ldots, u_n)$. Let U be the set of all games.

Let $p : A \rightarrow \Re$ be a probability distribution over A, in other words $p(a) \geq 0$ for all $a \in A$ and $\sum_{a \in A} p(a) = 1$. Let P be the set of all probability distributions over A. For $B \subset A$, we let $p(B) = \sum_{a \in B} p(a)$ be the probability of event B. We write $A_{-i} = \times_{j \in N \setminus \{i\}} A_j$, and an element of A_{-i} is written a_{-i}.

Correlated equilibrium is defined by a set of inequalities.

Definition. Given $p \in P$ and $u \in U$, we say that p is a correlated equilibrium of u if

$$\sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u_i(b_i, a_{-i}) \text{ for all } a_i, b_i \in A_i, i \in N. \quad (IC)$$

Given $u \in U$, the set of correlated equilibria of u is written $CE(u) \subset P$. Given $p \in P$, the set of games for which p is a correlated equilibrium is written $CE^{-1}(p) \subset U$.

The traditional interpretation of correlated equilibrium is that there exists some device which sends each person i a message to play a_i, and the device is programmed to send these messages with probability distribution $p(a)$. The inequality IC, an "incentive compatibility" constraint, requires that if person i gets the message to play a_i, the expected payoff she gets from following this message and playing a_i, assuming that everyone else follows their own messages, is at least as great as the expected payoff she would get if she were to play b_i instead.

This paper uses a "statistical" interpretation. Say that p is the observed distribution of play in a game. Whenever person i played a_i, she could have played b_i instead; the definition simply requires that person i could not have gained by doing so. In this sense, the IC inequalities are minimal requirements for individual rationality, and can also be understood as "revealed preference" inequalities. How exactly the distribution p occurs is not specified, just as how a particular Nash equilibrium occurs is not specified in the definition of Nash equilibrium.

It is easy to see that $CE(u)$ and $CE^{-1}(p)$ are both convex sets. We know $CE(u) \neq \emptyset$ from Hart and Schmeidler (1989) and Nau and McCardle (1990), and also from the existence of Nash equilibria. We know $CE^{-1}(p) \neq \emptyset$ because any p is a correlated equilibrium of the trivial game u, where u is defined as $u_i(a) = 0$ for all $a \in A$.

A real-valued random variable $f : A \to \Re$ is a function from the set A to the real numbers. Given probability distribution p, the covariance of two random variables f and g is $cov(f, g) = E(fg) - E(f)E(g)$, where $E(f) = \sum_{a \in A} p(a)f(a)$, $E(g) = \sum_{a \in A} p(a)g(a)$, and $E(fg) = \sum_{a \in A} p(a)f(a)g(a)$. Similarly, given some set $B \subset A$ such that $p(B) > 0$, the covariance of f and g conditional on B is $cov(f, g|B) = E(fg|B) - E(f|B)E(g|B)$, where $E(f|B) = (\sum_{a \in B} p(a)f(a))/p(B)$, $E(g|B) = (\sum_{a \in B} p(a)g(a))/p(B)$, and $E(fg|B) = (\sum_{a \in B} p(a)f(a)g(a))/p(B)$. If $p(B) = 0$, we write $cov(f, g|B) = 0$ for convenience. For all random variables f, g^1, g^2 and $\alpha_1, \alpha_2 \in \Re$, we have $cov(f, \alpha^1 g^1 + \alpha^2 g^2) = \alpha^1 cov(f, g^1) + \alpha^2 cov(f, g^2)$; in other words, cov is a "linear operator." We define the function $\Box_i : A \to A_i$ as $\Box_i(a) = a_i$ and the function $\Box_{-i} : A \to A_{-i}$ as $\Box_{-i}(a) = a_{-i}$.

3. *Main result*

Our main result is in terms of a signed conditional covariance. In any correlated equilibrium of a game, conditional on person i playing either a_i or b_i, person i's action is nonnegatively correlated with the payoff difference between playing a_i and b_i. The proof is in the appendix.

Proposition. Say $p \in CE(u)$ and $a_i, b_i \in A_i$, where $a_i > b_i$. Then

$$cov(\Box_i, u_i(a_i, \Box_{-i}) - u_i(b_i, \Box_{-i}) | \{a_i, b_i\} \times A_{-i}) \geq 0.$$

This result is obtained by manipulating two IC constraints: the constraint that when person i plays a_i, she cannot do better by playing b_i, and the constraint that when person i plays b_i, she cannot do better by playing a_i. Note that the IC constraints are linear in p while the covariance in the Proposition is quadratic in p; in other words, it is not a linear restatement of the IC constraints.

To illustrate what the Proposition says, consider the following two person game.

	2	3	4
0	8, 0	0, 3	5, 5
1	0, 9	7, 7	6, 2

If we let $i = 1$, $a_i = 1$ and $b_i = 0$, the Proposition says that the two random variables $\Box_1$ and $u_1(1, \Box_2) - u_1(0, \Box_2)$ are nonnegatively correlated. These two random variables are shown below.

	2	3	4
0	0	0	0
1	1	1	1

$\Box_1$

	2	3	4
0	−8	7	1
1	−8	7	1

$u_1(1, \Box_2) - u_1(0, \Box_2)$

For these two random variables to have nonnegative covariance, it must be that roughly speaking, when $u_1(1, \Box_2) - u_1(0, \Box_2)$ is high, $\Box_1$ is high. Inspecting the two random variables above, this would mean very roughly that $p(0, 2)$ and $p(1, 3)$ are large in comparison with $p(0, 3)$ and $p(1, 2)$, for example.

4. 2×2 *games*

We first explore the implications of the Proposition for 2×2 games. In a "generic" 2×2 game, we can sign the covariance of the two players' actions in all correlated equilibria.

Corollary 1. Say $n = 2$ and $A_1 = A_2 = \{0, 1\}$. Say that $CE(u) \neq P$. Then either $cov(\Box_1, \Box_2) \geq 0$ for all $p \in CE(u)$ or $cov(\Box_1, \Box_2) \leq 0$ for all $p \in CE(u)$.

The proof is in the appendix, but it is easy to explain how it works. The Proposition says that $cov(\square_1, u_1(1, \square_2) - u_1(0, \square_2)) \geq 0$. Since in our 2×2 game $u_1(1, \square_2) - u_1(0, \square_2) = (u_1(1,1) - u_1(0,1))\square_2 + (u_1(1,0) - u_1(0,0))(1 - \square_2)$, we have $(u_1(1,1) - u_1(0,1) - u_1(1,0) + u_1(0,0))cov(\square_1, \square_2) \geq 0$. Thus it is possible to sign $cov(\square_1, \square_2)$. In other words, the Proposition signs the covariance between person 1's strategy and his payoff difference; in a 2×2 game, the payoff difference is a linear function of person 2's strategy, and thus it is possible to sign the covariance between person 1's strategy and person 2's strategy.

Consider the example games below. The first two, "chicken" and "battle of the sexes," have two pure strategy Nash equilibria and one mixed strategy Nash equilibrium, and the covariance between the players' strategies can be signed.

	0	1
0	3, 3	1, 4
1	4, 1	0, 0

$cov \leq 0$

	0	1
0	2, 1	0, 0
1	0, 0	1, 2

$cov \geq 0$

	0	1
0	1, 0	0, 1
1	0, 1	1, 0

$cov = 0$

	0	1
0	5, 0	5, 8
1	6, 4	6, 4

$cov = 0$

	0	1
0	5, 3	7, 3
1	5, 2	7, 2

$CE(u) = P$

In the third game, "matching pennies," the mixed strategy Nash equilibrium is also the unique correlated equilibrium, and hence the covariance is zero. In the fourth game, the correlated equilibria are those distributions which place weight only on $(1, 0)$ and $(1, 1)$, since for person 1 action 0 is strongly dominated. Since there is no variation in person 1's action, the covariance between their actions is zero. The fifth game is an example of the "degenerate" case when $CE(u) = P$, that is, every distribution is a correlated equilibrium.

If we consider the question of identification, it turns out that a signed covariance is sufficient to identify a game's pure strategy Nash equilibria. In other words, if one observes a nonzero covariance between two people's actions in a 2×2 game, we need not care if their actions result from pure strategy Nash equilibria, mixed strategy Nash equilibria, correlated equilibria, or a mixture of all of these. Without any prior knowledge of what game they are playing, the signed covariance itself is enough to locate pure Nash equilibria of the game.

Corollary 2. Say $n = 2$, $A_1 = A_2 = \{0, 1\}$, and $p \in CE(u)$. If $cov(\square_1, \square_2) > 0$, then $(0, 0)$ and $(1, 1)$ are Nash equilibria of u. If $cov(\square_1, \square_2) < 0$, then $(0, 1)$ and $(1, 0)$ are Nash equilibria of u.

The proof of this is also simple: from the Proposition, we have $(u_1(1,1) - u_1(0,1) - u_1(1,0) + u_1(0,0))cov(\square_1, \square_2) \geq 0$ and thus if $cov(\square_1, \square_2) > 0$, we have $u_1(1,1) - u_1(0,1) -$

$u_1(1,0) + u_1(0,0) \geq 0$. It cannot be that person 1 has a strongly dominated strategy (because then $cov(\Box_1, \Box_2) = 0$ in any correlated equilibrium) and thus $u_1(1,1) \geq u_1(0,1)$ and $u_1(0,0) \geq u_1(1,0)$. We show $u_2(1,1) \geq u_2(1,0)$ and $u_2(0,0) \geq u_2(0,1)$ similarly.

After identifying a game to the extent possible given observations, it is natural to then make a prediction based on what has been learned. In a 2×2 game, this could not be simpler. If one observes for example positive covariance, in any such game consistent with this observation, in any correlated equilibrium of this game, one must have nonnegative covariance (as long as the game is "generic" in the sense that not everything is a correlated equilibrium). The following result follows directly from Corollary 2.

Corollary 3. Say $n = 2$, $A_1 = A_2 = \{0, 1\}$, and $p, p' \in CE(u)$. Say $CE(u) \neq P$. If $cov(\Box_1, \Box_2) > 0$ given p, then $cov(\Box_1, \Box_2) \geq 0$ given p'. If $cov(\Box_1, \Box_2) < 0$ given p, then $cov(\Box_1, \Box_2) \leq 0$ given p'.

In other words, if one observes signed covariance in observed play of any 2×2 game, one can immediately predict that in any correlated equilibrium, the covariance cannot have the opposite sign. One can make this prediction without knowing anything further about the game itself (other than assuming that the game is "generic" in the sense that not everything is a correlated equilibrium).

5. Linear-quadratic payoffs

Payoff functions which are linear-quadratic in people's strategies are often found in applications: one common example is a Cournot oligopoly game with a linear demand function and quadratic costs (see Liu 1996 and Yi 1997 on the uniqueness of correlated equilibria in Cournot oligopoly games and Neyman 1997 on potential games generally). Any game in which best response functions are linear (see for example Manski 1995, p. 116) is naturally represented with linear-quadratic payoff functions. When person i's payoffs are linear-quadratic (actually, when they satisfy a somewhat weaker condition), we can sign the weighted sum of covariances between person i's strategy and everyone else's.

Corollary 4. Say that u_i satisfies the condition that $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) =$ $f(a_i, b_i) \sum_{j \in N \setminus \{i\}} c_{ij} a_j + g(a_i, b_i)$, where $c_{ij} \in \Re$, f and g are functions of only a_i and b_i, and $f(a_i, b_i) > 0$ when $a_i > b_i$. Say $p \in CE(u)$. Then $\sum_{j \in N \setminus \{i\}} c_{ij}\, cov(\Box_i, \Box_j) \geq 0$.

For example, say that $n = 2$ and u_1 is linear-quadratic: $u_1(a_1, a_2) = k_{12}a_1a_2 + k_{11}(a_1)^2 + k_{22}(a_2)^2 + k_1a_1 + k_2a_2 + k_0$. Here k_{12} is an "interaction term" which affects how person 1's payoffs change with person 2's action. It is easy to see that this payoff function satisfies the condition in Corollary 4: set $c_{12} = k_{12}$, $f(a_1, b_1) = a_1 - b_1$ and $g(a_1, b_1) = k_{11}((a_1)^2 - (b_1)^2) + k_1(a_1 - b_1)$. Corollary 4 says that $k_{12}\, cov(\Box_1, \Box_2) \geq 0$. As long as one person's payoff function is linear-quadratic and $k_{12} \neq 0$, we can sign the covariance of the two persons' strategies in all correlated equilibria. Similarly, if we observe a positive covariance in the data, if we asssume that one person's payoffs are linear-quadratic, we can conclude that $k_{12} \geq 0$. Finally, if we observe a signed covariance in the observed play of any two-person game in which one person's payoffs are linear-quadratic, we can predict that the covariance in any correlated equilibrium of the game must have the same sign. We make this prediction knowing nothing else about the game, other than assuming that $k_{12} \neq 0$.

The proof is in the appendix but again it is easy to explain how it works. The Proposition signs the covariance between person i's strategy and his utility difference. When the condition in Corollary 4 is satisfied, the utility difference is a weighted sum of the various a_{-i}, and thus we can sign the weighted sum of covariances between person i's strategy and the various a_{-i}.

For an example which is not linear-quadratic, say that $n = 3$ and $u_1(a_1, a_2, a_3) = 5(a_1)^{1/2}a_2 - 3(a_1)^{1/2}a_3 + 4a_2a_3 + 6(a_2 - a_3)^2 - 7(a_1)^{3/2}$. By Corollary 4, we have $5cov(\Box_1, \Box_2) - 3cov(\Box_1, \Box_3) \geq 0$.

Sometimes Corollary 4 allows us to sign all covariances. For example, say that $n = 3$, $u_1(a_1, a_2, a_3) = a_1a_3 - a_1a_2 - (a_1)^2$, $u_2(a_1, a_2, a_3) = a_1a_2 - a_2a_3 - (a_2)^2$, and $u_3(a_1, a_2, a_3) = -(a_2 - a_3)^2$. By Corollary 4, we know that $cov(\Box_1, \Box_3) - cov(\Box_1, \Box_2) \geq 0$, $cov(\Box_1, \Box_2) - cov(\Box_2, \Box_3) \geq 0$, and $2cov(\Box_2, \Box_3) \geq 0$. Thus we can conclude that $cov(\Box_1, \Box_2)$, $cov(\Box_1, \Box_3)$, and $cov(\Box_2, \Box_3)$ are all nonnegative in any correlated equilibrium of this game.

6. Another example

The following "Three Player Matching Pennies Game" below is studied experimentally by Moreno and Wooders (1998).

	0	1
0	$1, 1, -2$	$-1, -1, 2$
1	$-1, -1, 2$	$-1, -1, 2$

0

	0	1
0	$-1, -1, 2$	$-1, -1, 2$
1	$-1, -1, 2$	$1, 1, -2$

1

The Proposition says that $cov(\Box_1, u_1(1, \Box_2, \Box_3) - u_1(0, \Box_2, \Box_3)) \geq 0$ in any correlated equilibrium. In this game, $u_1(1, \Box_2, \Box_3) - u_1(0, \Box_2, \Box_3) = -2(1-\Box_2)(1-\Box_3) + 2\Box_2\Box_3 = -2 + 2\Box_2 + 2\Box_3$. Since -2 is a constant, we have $cov(\Box_1, 2\Box_2 + 2\Box_3) \geq 0$ and thus $cov(\Box_1, \Box_2) + cov(\Box_1, \Box_3) \geq 0$. Similarly, we find that $cov(\Box_1, \Box_2) + cov(\Box_2, \Box_3) \geq 0$. We also know that $cov(\Box_3, u_3(\Box_1, \Box_2, 1) - u_3(\Box_1, \Box_2, 0)) \geq 0$ and that $u_3(\Box_1, \Box_2, 1) - u_3(\Box_1, \Box_2, 0) = 4(1 - \Box_1)(1 - \Box_2) - 4\Box_1\Box_2 = 4 - 4\Box_1 - 4\Box_2$. Thus $cov(\Box_1, \Box_3) + cov(\Box_2, \Box_3) \leq 0$. So we have three inequalities on the three covariances $cov(\Box_1, \Box_2)$, $cov(\Box_1, \Box_3)$, and $cov(\Box_2, \Box_3)$. From these inequalities, we conclude that $cov(\Box_1, \Box_2)$ is nonnegative and either $cov(\Box_1, \Box_3)$ or $cov(\Box_2, \Box_3)$ or both are nonpositive for all correlated equilibria of this game.

7. *Linear combinations of games*

Sometimes it is convenient to express payoffs in a game as the linear combination of payoffs from several simpler games. For example, DeNardo (1995) surveys expert and student preferences over whether the United States and Soviet Union should build weapons systems such as the MX missile, and finds that the great variety of preferences, elicited from questionaires, are understandable as convex combinations of certain "strategic extremes" such as the "Pure Dove" and the "Strong Hawk," shown below.

	SU builds	SU doesn't
US builds	1	1
US doesn't	1	4

Pure Dove

	SU builds	SU doesn't
US builds	3	4
US doesn't	1	2

Strong Hawk

Here the Pure Dove prefers for neither side to build the weapon, and any side building the weapon is equally bad. For the Strong Hawk, US superiority is most preferred, both having the weapon is second best, and the worst is for the US to not have the weapon while the Soviet Union does. DeNardo finds that roughly 22 percent of student preferences can be represented as convex combinations of these two extremes. Let α be the weight given to Pure Dove and $1 - \alpha$ be the weight given to Strong Hawk, where $\alpha \in [0, 1]$. Say the

US is person 1 and the Soviet Union is person 2, and that the preferences above are the US's preferences. Say that building is strategy 1 and not building is strategy 0. In Pure Dove, $u_1(1, \Box_2) - u_1(0, \Box_2) = -3(1 - \Box_2)$. In Strong Hawk, $u_1(1, \Box_2) - u_1(0, \Box_2) = 2$. Hence in the game which is a convex combination of Pure Dove and Strong Hawk, we have $cov(\Box_1, \Box_2) = \alpha cov(\Box_1, -3(1-\Box_2)) + (1-\alpha) cov(\Box_1, 2) \geq 0$. Thus we get $3\alpha cov(\Box_1, \Box_2) \geq 0$. If $\alpha > 0$, we know $cov(\Box_1, \Box_2) \geq 0$. If $\alpha = 0$, in any correlated equilibrium, the US always builds and hence $cov(\Box_1, \Box_2) = 0$. Regardless of what α is, we can conclude that the US and SU actions are nonnegatively correlated.

For another example, say that we observe people playing a 2×2 game. We do not know exactly which game they are playing; they might be playing either chicken, battle of the sexes, or matching pennies, or some mixture of the three. Assume that the game they are playing is a convex combination of these three, with chicken having weight α, battle of sexes having weight β, and matching pennies having weight $1 - \alpha - \beta$, as shown below.

	0	1		0	1		0	1
0	3, 3	1, 4	0	2, 1	0, 0	0	1, 0	0, 1
1	4, 1	0, 0	1	0, 0	1, 2	1	0, 1	1, 0
	α			β			$1 - \alpha - \beta$	

From the Proposition, we have $\alpha cov(\Box_1, 1 - 2\Box_2)) + \beta cov(\Box_1, -2 + 3\Box_2) + (1 - \alpha - \beta) cov(\Box_1, -1 + 2\Box_2) \geq 0$. Hence $(2 - 4\alpha + \beta) cov(\Box_1, \Box_2) \geq 0$. From the Proposition we also have $\alpha cov(\Box_2, 1 - 2\Box_1)) + \beta cov(\Box_2, -1 + 3\Box_1) + (1 - \alpha - \beta) cov(\Box_2, 1 - 2\Box_1) \geq 0$. Hence $(5\beta - 2) cov(\Box_1, \Box_2) \geq 0$. Thus if we observe a positive covariance between their strategies, we can conclude that $\beta \geq 2/5$ and $\alpha \leq 3/5$. If we observe a negative covariance, we can conclude that $\beta \leq 2/5$ and $\alpha \geq 3/5$. In other words, a positive covariance indicates that the battle of the sexes "component" is relatively large, while a negative covariance indicates that it is relatively small.

8. Likelihood ratios and supermodular games

We can also think about correlated equilibrium in terms of "likelihood ratios." One might expect that in a correlated equilibrium, the ratio $p(a_i, \cdot)/p(b_i, \cdot)$ should be more or less increasing in the payoff difference $u_i(a_i, \cdot) - u_i(b_i, \cdot)$. In other words, a_i should be played more often relative to b_i when the payoff difference between a_i and b_i is higher. We cannot make this strong statement (because it is not true), but we can make a weaker one. Corollary

5 says that the ratio cannot be strictly decreasing in the payoff difference: it cannot be that a_i is always played less often relative to b_i when the payoff difference between a_i and b_i is higher.

Corollary 5. Say that $p \in CE(u)$ and $a_i, b_i \in A_i$. Say that $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})$ is not constant in a_{-i} and that $p(b_i, a_{-i}) > 0$ for $a_{-i} \in A_{-i}$. The following statement is not true: For all $a_{-i}, b_{-i} \in A_{-i}$,

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) > u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}) \Rightarrow p(a_i, a_{-i})/p(b_i, a_{-i}) < p(a_i, b_{-i})/p(b_i, b_{-i}).$$

The proof is in the appendix, but it is easy to explain how it works. Say that $a_i > b_i$. By the Proposition, we know that person i's strategy and his payoff difference are nonnegatively correlated. If a_i is always played less often relative to b_i when the payoff difference is higher, then we would have a negative correlation.

In a two-person game, if $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})$ strictly increases in a_{-i} for all $a_i > b_i$, person i's payoffs are called strictly supermodular. If $p(a_i, a_{-i})/p(b_i, a_{-i})$ strictly increases in a_{-i} for all $a_i > b_i$, then p is called strictly totally positive of order 2 (Karlin and Rinott 1980a; Milgrom and Weber 1982 use the term "affiliated"). If $p(a_i, a_{-i})/p(b_i, a_{-i})$ strictly decreases in a_{-i} for all $a_i > b_i$, then p is called strictly reverse rule of order 2 (Karlin and Rinott 1980b). From Corollary 5, we know that in a two-person game in which one person's payoffs are strictly supermodular, a correlated equilibrium might not be strictly totally positive of order 2, but it cannot be strictly reverse rule of order 2.

9. Strictly competitive games

When one person's payoff is strictly decreasing in the other person's payoff, one might expect that the only correlated equilibria are those in which people's strategies are independently distributed (that is, pure strategy or mixed strategy Nash equilibria) because there are no joint "gains from correlation." For now we can prove this for "generic" $2 \times m$ games. The proof is in the appendix.

Corollary 6. Say that $n = 2$, $A_1 = \{0, 1\}$ and that $u_1(a) = u_1(b) \Leftrightarrow u_2(a) = u_2(b)$ and $u_1(a) > u_1(b) \Leftrightarrow u_2(a) < u_2(b)$. Say that there are no "duplicated strategies": there do not exist $a_2, b_2 \in A_2$ such that $a_2 \neq b_2$ and $u_2(0, a_2) = u_2(0, b_2)$ and $u_2(1, a_2) = u_2(1, b_2)$. Say $p \in CE(u)$. Then $\square_1$ and $\square_2$ are independently distributed given p.

Under the additional assumption that the game is zero-sum, it has been shown the set of correlated equilibrium payoffs is equal to the set of Nash equilibrium payoffs (Viossat 2003; see also Forges 1990 and Rosenthal 1974) and in this sense, there are no nontrivial correlated equilibria of a zero-sum game. Corollary 6 applies in the slightly more general context of strictly competitive games.

10. Local interaction games

A local interaction game can be thought of as each person playing the same 2×2 game with each of his neighbors. We let $A_i = \{0, 1\}$ for all $i \in N$ and for each $i \in N$, say that there is a set $N(i) \subset N$ which represents person i's neighbors (we assume $i \notin N(i)$). Payoffs are defined as $u_i(a) = \sum_{j \in N(i)} v_i(a_i, a_j)$.

The Proposition says that $\sum_{j \in N \setminus \{i\}} cov(\square_i, v_i(1, \square_j) - v_i(0, \square_j)) \geq 0$. But we know $v_i(1, \square_j) - v_i(0, \square_j) = (v_i(1,0) - v_i(0,0))(1 - \square_j) + (v_i(1,1) - v_i(0,1))\square_j = v_i(1,0) - v_i(0,0) + [v_i(0,0) - v_i(1,0) + v_i(1,1) - v_i(0,1)]\square_j$. Since $v_i(1,0) - v_i(0,0)$ is a constant, we have the following.

Corollary 7. Say $A_i = \{0, 1\}$ for all $i \in N$ and $u_i(a) = \sum_{j \in N(i)} v_i(a_i, a_j)$, where $N(i) \subset N$ and $i \notin N(i)$. Say $p \in CE(u)$. Then $[v_i(0,0) - v_i(1,0) + v_i(1,1) - v_i(0,1)] cov(\square_i, \sum_{j \in N(i)} \square_j) \geq 0$.

In any local interaction game, given the neighborhood $N(i)$ and the payoffs v_i, we can therefore sign the covariance between a person's action and the sum of his neighbors' actions, as long as $v_i(0,0) - v_i(1,0) + v_i(1,1) - v_i(0,1) \neq 0$. For example, if v_i is a coordination game, it must be that person i's action is nonnegatively correlated with the sum of her neighbors' actions. Going in the other direction, given observed actions and the neighborhood $N(i)$ of person i, we can sign $v_i(0,0) - v_i(1,0) + v_i(1,1) - v_i(0,1)$. Given observed actions and payoffs v_i, we can identify possible neighborhoods $N(i)$.

The equilibria and best response dynamics of local interaction games are a current subject of research (see for example Young 1998 and Morris 2000); and it would seem that considering their correlated equilibria would only complicate things further. If we think in terms of the signed covariances which are implied by correlated equilibrium, however, we have a simple and intuitive conclusion.

11. A spatial data example

On October 1, 3, and 5, 2001, I collected data on whether people in census tract 7016.01 (in Santa Monica, California) displayed flags on their residences, as shown in Figure 1. A plus sign indicates a residence which displays a United States flag (or some sort of decoration which includes the colors red, white, and blue); a dot indicates a residence which does not. There are 1174 total residences in the data set, which is available from the author. The residences in this census tract are primarily single-family homes, although 93 buildings in my data set are multi-unit buildings such as townhouses, duplexes, or apartment buildings. A data point here is an individual building; for example, when a flag appears on an apartment building or other multiunit building, the entire building is counted as displaying a flag and no attempt is made to figure out which apartment in the building is displaying the flag and which ones are not. Only residential buildings are included. According to the 2000 US Census, 3957 people live in this census tract and there are a total of 1863 housing units.

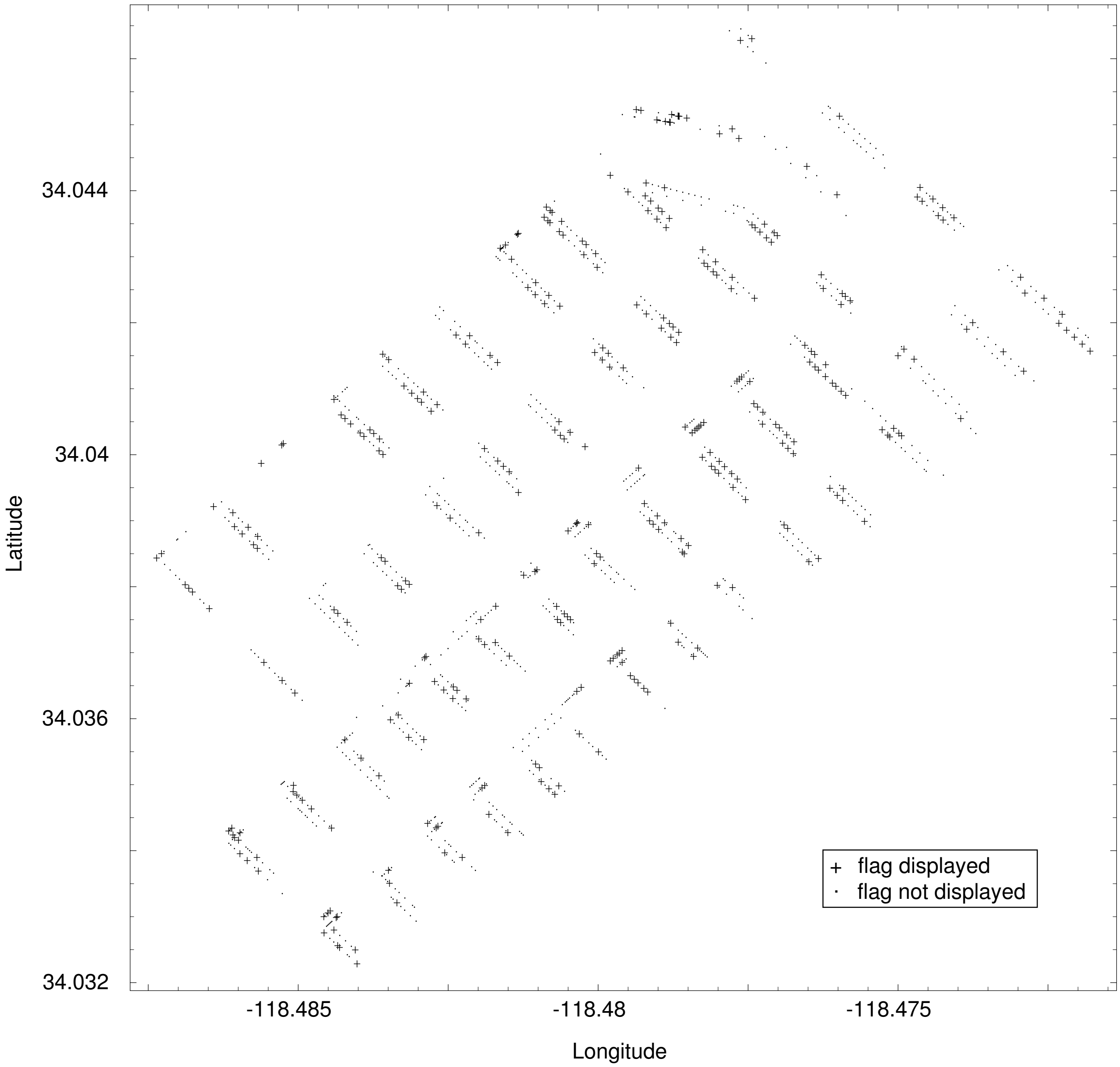

Figure 1. Flag display in census tract 7016.01
(Santa Monica, California), October 1, 3, 5, 2001

In my data, 362 of the 1174 residences had flags displayed (30.8 percent). Inspecting Figure 1, it seems that a person's choice of whether to display a flag depends on whether her neighbors display a flag; for example, there are some blocks in which nearly everyone displays a flag, which would be unlikely if people's decisions were independent.

Thus we might say that the people here are playing a game in which a person's payoff from displaying a flag depends on how many of her neighbors also display flags. If we say that putting up a flag is strategy 1 and not putting up one is strategy 0, then this is a local interaction game as described earlier, with payoffs $v(0,0), v(0,1), v(1,0), v(1,1)$ (assume these payoffs are the same for everyone). Let $N(i)$, the neighborhood of i, be the houses on the same block adjacent to i. In our data, 947 of the 1174 residences have two neighbors in this sense, 220 have one neighbor, and 7 have no neighbors. We find that $cov(\square_i, \sum_{j \in N(i)} \square_j) = (250/1174) - (362/1174)(655/1174) \approx 0.0409$; in other words, the covariance between a person's action and the actions of his neighbors is positive. Hence from Corollary 6, we know $v(0,0) - v(1,0) + v(1,1) - v(0,1) \geq 0$. Since both strategy 1 and strategy 0 are observed, we assume that neither is strongly dominated, and hence we can conclude that $v(0,0) \geq v(1,0)$ and $v(1,1) \geq v(0,1)$, or in other words, v is a coordination game with two pure Nash equilibria $(0,0)$ and $(1,1)$.

If we want to identify $v(0,0), v(0,1), v(1,0), v(1,1)$ more precisely, we can directly use the IC inequalities in the definition of correlated equilibrium. Of the residences which put up flags, on average $250/362 \approx 0.691$ of their neighbors also put up flags and $405/362 \approx 1.119$ of their neighbors do not put up flags. Hence we have the inequality $(250/362)v(1,1) + (405/362)v(1,0) \geq (250/362)v(0,1) + (405/362)v(0,0)$. Of the residences which do not put up flags, on average $405/812 \approx 0.499$ of their neighbors put up flags and $1054/812 \approx 1.298$ of their neighbors do not put up flags. Hence we have the inequality $(405/812)v(0,1) + (1054/812)v(0,0) \geq (405/812)v(1,1) + (1054/812)v(1,0)$. We can normalize $v(0,1) = v(1,0) = 0$; assuming that $v(0,0) \neq 0$, we can also normalize $v(0,0) = 1$. We thus have $v(1,1) \in [405/250, 1054/450] \approx [1.620, 2.342]$.

We might suppose a more complicated model; for example a person's immediate neighbors might affect her payoff more than people who live two houses away. Say that person i has immediate neighbors $N(i)$ and peripheral neighbors $NN(i)$. Say that a person gets payoffs $v(0,0), v(0,1), v(1,0), v(1,1)$ from immediate neighbors and payoffs $vv(0,0), vv(0,1), vv(1,0), vv(1,1)$ from peripheral neighbors. Assume that $v(0,1) = v(1,0) = vv(0,1) = vv(1,0) = 0$. So if a person puts up a flag and one of her immediate neighbors and two of her peripheral neighbors put up flags, she gets payoff $v(1,1) + 2vv(1,1)$ for example. From the Proposition, we know that $(v(0,0) + v(1,1))cov(\square_i, \sum_{j \in N(i)} \square_j) + (vv(0,0) + vv(1,1))cov(\square_i, \sum_{j \in NN(i)} \square_j) \geq 0$. Let $N(i)$ again be the houses on the same block adjacent to i and let $NN(i)$ be the houses on the same block two houses away from i on either side. In the data, we have $cov(\square_i, \sum_{j \in N(i)} \square_j) \approx 0.0409$ and $cov(\square_i, \sum_{j \in NN(i)} \square_j) \approx 0.0278$. Thus

$0.0409(v(0,0)+v(1,1))+0.0278(vv(0,0)+vv(1,1)) \geq 0$. This imposes some weak restrictions on v and vv; for example, it cannot be that both v and vv are "anti-coordination" games (games in which $(0,1)$ and $(1,0)$ are the pure Nash equilibria).

For further restrictions on v and vv, again we can directly use the IC inequalities. Of the residences which put up flags, on average $250/362 \approx 0.691$ of their immediate neighbors also put up flags and $405/362 \approx 1.119$ of their immediate neighbors do not put up flags. On average, $218/362 \approx 0.602$ of their peripheral neighbors put up flags and $383/362 \approx 1.058$ of their peripheral neighbors do not up flags. Hence we have the inequality $(250/362)v(1,1) + (218/362)vv(1,1) \geq (405/362)v(0,0) + (383/362)vv(0,0)$. Of the residences which do not put up flags, on average $405/812 \approx 0.499$ of their immediate neighbors put up flags and $1054/812 \approx 1.298$ of their neighbors do not put up flags. On average, $383/812 \approx 0.472$ of their peripheral neighbors put up flags and $910/812 \approx 1.121$ of their peripheral neighbors do not put up flags. Hence we have the inequality $(1054/812)v(0,0) + (910/812)vv(0,0) \geq (405/812)v(1,1) + (383/812)vv(1,1)$.

We thus have four variables and two inequalities. Since there are more "degrees of freedom" in this more complicated model, the data provide fewer restrictions. More observations would provide more inequalities and tighter identification. For now, if we simplify matters by assuming $v(0,0) = vv(0,0) = 1$, then we have $250v(1,1) + 218vv(1,1) \geq 788$ and $405v(1,1) + 383vv(1,1) \leq 1964$. The feasible region for $v(1,1)$ and $vv(1,1)$ is shown in Figure 2 below.

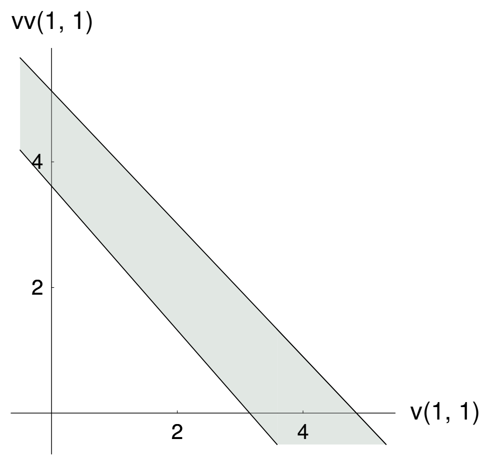

Figure 2. Feasible region for $v(1,1)$ and $vv(1,1)$, assuming $v(0,0) = vv(0,0) = 1$

The analysis here is not meant to be conclusive (there are standard approaches for analyzing spatial data with which our analysis might be compared), but rather an illustration of how the results in this paper can be applied to real-world data in a very direct and simple way. At some point, we might want to introduce assumptions about the distributions of errors and individual heterogeneity, but we can make some meaningful conclusions without doing so. For example, we conclude that the data is consistent with a local interaction game which has pure Nash equilibria $(0, 0)$ and $(1, 1)$ and can identify the magnitude of $v(1, 1)$ relative to $v(0, 0)$. Note that if we assume Nash equilibrium instead of correlated equilibrium, we cannot identify the magnitude of $v(1, 1)$ because any positive $v(1, 1)$ is consistent with $(1, 1)$ being a Nash equilibrium. Instead of adding extraneous statistical elements to a game, our approach exploits the statistical relationships inherent in the game itself.

Appendix

To prove the Proposition, we first define $x_i(a,b)$ and derive two lemmas. Given $p \in P$ and $a, b \in A$, define $x_i(a,b) = p(a)p(b) - p(a_i, b_{-i})p(b_i, a_{-i})$. Lemma 1 says that correlated equilibrium implies linear inequalities on $x_i(a,b)$.

Lemma 1. If $p \in CE(u)$, then $\sum_{a_{-i} \in A_{-i}} (u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}))x_i(a,b) \geq 0$.

Proof. We write IC as $\sum_{a_{-i}} p(a_i, a_{-i})(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) \geq 0$. Multiplying both sides by $p(b_i, b_{-i})$, we have $\sum_{a_{-i}} p(a_i, a_{-i})p(b_i, b_{-i})(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) \geq 0$. We call this inequality $(*)$. Similarly, we have the IC inequality $\sum_{a_{-i}} p(b_i, a_{-i})(u_i(b_i, a_{-i}) - u_i(a_i, a_{-i})) \geq 0$. Multiplying both sides by $p(a_i, b_{-i})$, we have $\sum_{a_{-i}} p(a_i, b_{-i})p(b_i, a_{-i})(u_i(b_i, a_{-i}) - u_i(a_i, a_{-i})) \geq 0$. We call this inequality $(**)$. Add the inequalities $(*)$ and $(**)$ together and we are done. ■

Lemma 2 says that the covariance of two random variables is a linear function of the $x_i(a,b)$. Lemma 2 is well known (see for example C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre 1971), but we state and prove it here for the sake of completeness. Let $\mathcal{A}_i$ and $\mathcal{A}_{-i}$ be partitions of A, where $\mathcal{A}_i = \{\{a_i\} \times A_{-i}\}_{a_i \in A_i}$ and $\mathcal{A}_{-i} = \{A_i \times \{a_{-i}\}\}_{a_{-i} \in A_{-i}}$. Thus a function $f : A \to \Re$ is measurable with respect to $\mathcal{A}_i$ if $a_i = b_i \Rightarrow f(a) = f(b)$ and f is measurable with respect to $\mathcal{A}_{-i}$ if $a_{-i} = b_{-i} \Rightarrow f(a) = f(b)$.

Lemma 2. Say that $f : A \to \Re$ is measurable with respect to $\mathcal{A}_i$ and $g : A \to \Re$ is measurable with respect to $\mathcal{A}_{-i}$. Say that $B = B_i \times B_{-i}$, where $B_i \subset A_i$, $B_{-i} \subset A_{-i}$, and $p(B) > 0$. Then $cov(f, g|B) = (1/2p(B)^2) \sum_{a,b \in B: a_i > b_i} (f(a) - f(b))(g(a) - g(b))x_i(a,b)$.

Proof. Let $k = \sum_{a,b \in B} (f(a) - f(b))(g(a) - g(b))p(a)p(b)$. Expanding the product, we have $k = \sum f(a)g(a)p(a)p(b) - \sum f(a)g(b)p(a)p(b) - \sum f(b)g(a)p(a)p(b) + \sum f(b)g(b)p(a)p(b) = p(B)^2[E(fg|B) - E(f|B)E(g|B) - E(f|B)E(g|B) + E(fg|B)] = 2p(B)^2 cov(f, g|B)$. We also have $k = (\sum_{a,b \in B: a_i > b_i} + \sum_{a,b \in B: a_i < b_i} + \sum_{a,b \in B: a_i = b_i})(f(a) - f(b))(g(a) - g(b))p(a)p(b)$. The third sum is zero because $f(a) - f(b) = 0$ when $a_i = b_i$. If we let $c = (b_i, a_{-i})$ and $d = (a_i, b_{-i})$, then the second sum can be written as $\sum_{c,d \in B: c_i > d_i} -(f(c) - f(d))(g(c) - g(d))p(d_i, c_{-i})p(c_i, d_{-i})$, because $B = B_i \times B_{-i}$, f is measurable with respect to $\mathcal{A}_i$, and g is measurable with respect to $\mathcal{A}_{-i}$. If we change variables again and let $a = c$ and $b = d$, the second sum can be written as $\sum_{a,b \in B: a_i > b_i} (f(a) - f(b))(g(a) - g(b))(-p(b_i, a_{-i})p(a_i, b_{-i}))$. Thus $k = \sum_{a,b \in B: a_i > b_i} (f(a) - f(b))(g(a) - g(b))(p(a)p(b) - p(b_i, a_{-i})p(a_i, b_{-i}))$ and we are done. ■

Proposition. Say $p \in CE(u)$ and $a_i, b_i \in A_i$, where $a_i > b_i$. Then

$$cov(\square_i, u_i(a_i, \square_{-i}) - u_i(b_i, \square_{-i}) | \{a_i, b_i\} \times A_{-i}) \geq 0.$$

Proof. Lemma 2 says $cov(\square_i, u_i(a_i, \square_{-i}) - u_i(b_i, \square_{-i}) | \{a_i, b_i\} \times A_{-i}) = (a_i - b_i)(1/2p(\{a_i, b_i\} \times A_{-i})^2) \sum_{a_{-i}, b_{-i} \in A_{-i}} [(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) - (u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}))] x_i(a, b)$. Since $a_i - b_i > 0$, this has the same sign as $\sum_{a_{-i}, b_{-i} \in A_{-i}} [(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) - (u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}))] x_i(a, b)$. So it suffices to show that this is nonnegative.

From Lemma 1, we know that $\sum_{a_{-i} \in A_{-i}} (u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) x_i(a, b) \geq 0$. Hence $\sum_{a_{-i}, b_{-i} \in A_{-i}} (u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) x_i(a, b) \geq 0$. Call this inequality $(*)$. From the definition of $x_i(a, b)$, we have $x_i(a, b) = -x_i((a_i, b_{-i}), (b_i, a_{-i}))$. Hence $\sum_{a_{-i}, b_{-i} \in A_{-i}} -(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) x_i((a_i, b_{-i}), (b_i, a_{-i})) \geq 0$. If we change variables and let $b_{-i} = a_{-i}$ and $a_{-i} = b_{-i}$, we have $\sum_{a_{-i}, b_{-i} \in A_{-i}} -(u_i(a_i, b_{-i}) - u_i(b_i, b_{-i})) x_i(a, b) \geq 0$. Call this inequality $(**)$. Add $(*)$ and $(**)$ together and we are done. ■

Corollary 1. Say $n = 2$ and $A_1 = A_2 = \{0, 1\}$. Say that $CE(u) \neq P$. Then either $cov(\square_1, \square_2) \geq 0$ for all $p \in CE(u)$ or $cov(\square_1, \square_2) \leq 0$ for all $p \in CE(u)$.

Proof. By the proposition, we know that $cov(\square_1, u_1(1, \square_2) - u_1(0, \square_2) \geq 0$. We have $u_1(1, \square_2) - u_1(0, \square_2) = (u_1(1,1) - u_1(0,1))\square_2 + (u_1(1,0) - u_1(0,0))(1 - \square_2)$. Thus $(u_1(1,1) - u_1(0,1) - u_1(1,0) + u_1(0,0)) cov(\square_1, \square_2) \geq 0$. Similarly, we have $(u_2(1,1) - u_2(1,0) - u_2(0,1) + u_2(0,0)) cov(\square_1, \square_2) \geq 0$. If either $u_1(1,1) - u_1(0,1) - u_1(1,0) + u_1(0,0) \neq 0$ or $u_2(1,1) - u_2(1,0) - u_2(0,1) + u_2(0,0) \neq 0$, we are done. Say $u_1(1,1) - u_1(0,1) - u_1(1,0) + u_1(0,0) = 0$ and $u_2(1,1) - u_2(1,0) - u_2(0,1) + u_2(0,0) = 0$. If $u_1(1,1) > u_1(0,1)$, then $u_1(1,0) > u_1(0,0)$ and strategy 1 dominates strategy 0 for person 1, which implies that $cov(\square_1, \square_2) = 0$ for all $p \in CE(u)$. If for example $u_1(1,0) > u_1(0,0)$ or $u_2(0,0) > u_2(0,1)$, a similar argument can be made. Thus we are left with the case when $u_1(1,1) = u_1(0,1)$, $u_1(0,1) = u_1(0,0)$ and $u_2(1,1) = u_2(1,0)$, $u_2(0,1) = u_2(0,0)$, in which case $CE(u) = P$. ■

Corollary 4. Say that u_i satisfies the condition that $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = f(a_i, b_i) \sum_{j \in N \setminus \{i\}} c_{ij} a_j + g(a_i, b_i)$, where $c_{ij} \in \Re$, f and g are functions of only a_i and b_i, and $f(a_i, b_i) > 0$ when $a_i > b_i$. Say $p \in CE(u)$. Then $\sum_{j \in N \setminus \{i\}} c_{ij}\, cov(\Box_i, \Box_j) \geq 0$.

Proof. Say $a_i > b_i$. By the Proposition, we know that $cov(\Box_i, f(a_i, b_i) \sum_{j \in N \setminus \{i\}} c_{ij} \Box_j + g(a_i, b_i) | \{a_i, b_i\} \times A_{-i}) \geq 0$. Since $f(a_i, b_i)$ and $g(a_i, b_i)$ are constants with respect to $\Box_i$ and $f(a_i, b_i) > 0$, we have $cov(\Box_i, \sum_{j \in N \setminus \{i\}} c_{ij} \Box_j | \{a_i, b_i\} \times A_{-i}) \geq 0$. By Lemma 2, we have $cov(\Box_i, \sum_{j \in N \setminus \{i\}} c_{ij} \Box_j) = \sum_{a_i > b_i} p(\{a_i, b_i\} \times A_{-i}) cov(\Box_i, \sum_{j \in N \setminus \{i\}} c_{ij} \Box_j | \{a_i, b_i\} \times A_{-i})$ and so we are done. ■

Corollary 5. Say that $p \in CE(u)$ and $a_i, b_i \in A_i$. Say that $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})$ is not constant in a_{-i} and that $p(b_i, a_{-i}) > 0$ for $a_{-i} \in A_{-i}$. The following statement is not true: For all $a_{-i}, b_{-i} \in A_{-i}$,

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) > u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}) \Rightarrow p(a_i, a_{-i})/p(b_i, a_{-i}) < p(a_i, b_{-i})/p(b_i, b_{-i}).$$

Proof. Assume without loss of generality that $a_i > b_i$. From the Proposition and its proof, we know that $\sum_{a_{-i}, b_{-i} \in A_{-i}} [(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) - (u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}))] x_i(a, b) \geq 0$. If the statement in Corollary 5 is true, then $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) > u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}) \Rightarrow p(a_i, a_{-i})/p(b_i, a_{-i}) < p(a_i, b_{-i})/p(b_i, b_{-i}) \Rightarrow x_i(a, b) < 0$. Similarly, if the statement in Corollary 5 is true, then $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) < u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}) \Rightarrow x_i(a, b) > 0$. Since $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})$ is not constant in a_{-i}, it must be that $\sum_{a_{-i}, b_{-i} \in A_{-i}} [(u_i(a_i, a_{-i}) - u_i(b_i, a_{-i})) - (u_i(a_i, b_{-i}) - u_i(b_i, b_{-i}))] x_i(a, b) < 0$, a contradiction. ■

Corollary 6. Say that $n = 2$, $A_1 = \{0, 1\}$ and that $u_1(a) = u_1(b) \Leftrightarrow u_2(a) = u_2(b)$ and $u_1(a) > u_1(b) \Leftrightarrow u_2(a) < u_2(b)$. Say that there are no "duplicated strategies": there do not exist $a_2, b_2 \in A_2$ such that $a_2 \neq b_2$ and $u_2(0, a_2) = u_2(0, b_2)$ and $u_2(1, a_2) = u_2(1, b_2)$. Say $p \in CE(u)$. Then $\Box_1$ and $\Box_2$ are independently distributed given p.

Proof. Say $p \in CE(u)$. If $p(0, a_2) = p(1, a_2) = 0$, then $p' \in CE(u')$, where $u' : A_1 \times A_2' \to \Re$, $A_2' = A_2 \setminus \{a_2\}$, $u'(a_1, a_2) = u(a_1, a_2)$ and $p' : A_1 \times A_2' \to [0, 1]$ is defined as $p'(a_1, a_2) = p(a_1, a_2)$. If $\Box_1$ and $\Box_2$ are independent given p', it is easy to see that they are independent given p. So without loss of generality, we assume that for all $a_2 \in A_2$, either $p(0, a_2) > 0$ or $p(1, a_2) > 0$. Note that if a_2 strongly dominates b_2, then $p(0, a_2) = p(1, a_2) = 0$; hence none of person 2's strategies are strongly dominated. Also, we write $x(a,b)$ instead of $x_i(a,b)$ because $x_1(a,b) = x_2(a,b)$ when $n = 2$.

Let $L_2 = \{a_2 \in A_2 : u_1(0,a_2) < u_1(1,a_2)\}$ and $M_2 = \{a_2 \in A_2 : u_1(0,a_2) \geq u_1(1,a_2)\}$. If $M_2 = \emptyset$, then strategy 1 strongly dominates strategy 0 and hence $p(0,a_2) = 0$ for all $a_2 \in A_2$, and we are done. So assume that $M_2 \neq \emptyset$.

Show that $x((0,a_2),(1,b_2)) \geq 0$ for all $a_2 \in L_2$ and $b_2 \in M_2$. Let $a_2 \in L_2$ and $b_2 \in M_2$. Thus $u_2(0,a_2) > u_2(1,a_2)$ and $u_2(0,b_2) \leq u_2(1,b_2)$. If $u_2(0,a_2) \leq u_2(0,b_2)$ and $u_2(1,a_2) \geq u_2(1,b_2)$, we have $u_2(0,a_2) \leq u_2(0,b_2) \leq u_2(1,b_2) \leq u_2(1,a_2) < u_2(0,a_2)$, a contradiction. Hence we must have either $u_2(0,a_2) > u_2(0,b_2)$ or $u_2(1,a_2) < u_2(1,b_2)$ or both. By Lemma 1 we know that $(u_2(0,a_2) - u_2(0,b_2))x((0,a_2),(1,b_2)) \geq 0$. Thus if $u_2(0,a_2) > u_2(0,b_2)$, we know $x((0,a_2),(1,b_2)) \geq 0$. Also by Lemma 1, we know that $(u_2(1,a_2) - u_2(1,b_2))x((1,a_2),(0,b_2)) \geq 0$. Thus if $u_2(1,a_2) < u_2(1,b_2)$, we have $x((1,a_2),(0,b_2)) \leq 0$ and thus $x((0,a_2),(1,b_2)) \geq 0$.

Let $c_2 \in argmin_{a_2 \in M_2} u_1(0,a_2) - u_1(1,a_2)$. Show that $x((0,c_2),(1,a_2)) \geq 0$ for all $a_2 \in M_2$. Let $a_2 \in M_2$. If $a_2 = c_2$, we have $x((0,c_2),(1,a_2)) = 0$, so assume that $a_2 \neq c_2$. We cannot have $u_2(0,c_2) = u_2(0,a_2)$ and $u_2(1,c_2) = u_2(1,a_2)$, because then c_2 and a_2 would be duplicate strategies. If $u_1(0,c_2) \geq u_1(0,a_2)$ and $u_1(1,c_2) \leq u_1(1,a_2)$, therefore at least one of the two inequalities is strict, and hence $u_1(0,c_2) - u_1(1,c_2) > u_1(0,a_2) - u_1(1,a_2)$, a contradiction of the definition of c_2. So it must be that either $u_2(0,c_2) > u_2(0,a_2)$ or $u_2(1,c_2) < u_2(1,a_2)$. By Lemma 1 we know that $(u_2(0,a_2) - u_2(0,c_2))x((0,a_2),(1,c_2)) \geq 0$. Thus if $u_2(0,c_2) > u_2(0,a_2)$, we have $x((0,a_2),(1,c_2)) \leq 0$, which means that $x((0,c_2),(1,a_2)) \geq 0$. Also by Lemma 1, we know that $(u_2(1,a_2) - u_2(1,c_2))x((1,a_2),(0,c_2)) \geq 0$. Thus if $u_2(1,c_2) < u_2(1,a_2)$, we have $x((1,a_2),(0,c_2)) \geq 0$, which means that $x((0,c_2),(1,a_2)) \geq 0$.

Show that if $a_2 \in M_2 \setminus \{c_2\}$, then $u_1(0,a_2) > u_1(1,a_2)$. If $a_2 \in M_2 \setminus \{c_2\}$, we know $u_1(0,a_2) \geq u_1(1,a_2)$. If $u_1(0,a_2) = u_1(1,a_2)$, we also have $u_1(1,c_2) = u_1(0,c_2)$ by the definition of c_2. Thus $u_2(1,a_2) = u_2(0,a_2)$ and $u_2(1,c_2) = u_2(0,c_2)$. If $u_2(1,a_2) \neq u_2(1,c_2)$, then either a_2 strongly dominates c_2 or vice versa, which cannot happen. If $u_2(1,a_2) = u_2(1,c_2)$, then a_2 and c_2 are duplicate strategies. Hence $u_1(0,a_2) > u_1(1,a_2)$.

By Lemma 1, we have $\sum_{a_2 \in A_2 \setminus \{c_2\}} (u_1(0,a_2) - u_1(1,a_2))x((0,a_2),(1,c_2)) \geq 0$ (since $x((0,c_2),(1,c_2)) = 0$). Thus we have $\sum_{a_2 \in L_2} (u_1(0,a_2) - u_1(1,a_2))x((0,a_2),(1,c_2)) + \sum_{a_2 \in M_2 \setminus \{c_2\}} (u_1(1,a_2) - u_1(0,a_2))x((0,c_2),(1,a_2)) \geq 0$. The left hand side of this inequality is linear in the variables $x((0,a_2),(1,c_2))$, where $a_2 \in L_2$, and $x((0,c_2),(1,a_2))$, where

$a_2 \in M_2 \setminus \{c_2\}$. All of these variables are positive or zero. All of the coefficients on these variables are negative. Hence all of the variables are zero.

Hence $x((0, a_2), (1, c_2)) = 0$ for all $a_2 \in A_2$. Hence $p(0, a_2)p(1, c_2) = p(0, c_2)p(1, a_2)$. By assumption we cannot have $p(0, c_2) = p(1, c_2) = 0$. If $p(0, c_2) = 0$ and $p(1, c_2) > 0$, then $p(0, a_2) = 0$ for all $a_2 \in A_2$, and we are done. Similarly, if $p(0, c_2) > 0$ and $p(1, c_2) = 0$, we are done. If $p(0, c_2) > 0$ and $p(1, c_2) > 0$, then $p(0, a_2)/p(1, a_2) = p(0, c_2)/p(1, c_2)$ for all $a_2 \in A_2$, and we are done. ■

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